SLOW BLOW-UP SOLUTIONS FOR THE $H^1(\mathbb{R}^3)$ CRITICAL FOCUSING SEMI-LINEAR WAVE EQUATION

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ABSTRACT. Given $\nu > \frac{1}{2}$ and $\delta > 0$ arbitrary, we prove the existence of energy solutions of

$$\partial_{tt}u - \Delta u - u^5 = 0 \tag{0.1}$$

in $\mathbb{R}^{3+1}$ that blow up exactly at $r = t = 0$ as $t \to 0-$. These solutions are radial and of the form $u = \lambda(t)^{\frac{1}{2}} W(\lambda(t)r) + \eta(r,t)$ inside the cone $r \le t$, where $\lambda(t) = t^{-1-\nu}$, $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ is the stationary solution of (0.1), and η is a radiation term with

$$\int_{[r\le t]} \left(|\nabla \eta(x,t)|^2 + |\eta_t(x,t)|^2 + |\eta(x,t)|^6\right) dx \to 0, \qquad t \to 0$$

Outside of the light-cone there is the energy bound

$$\int_{[r>t]} \left(|\nabla u(x,t)|^2 + |u_t(x,t)|^2 + |u(x,t)|^6\right) dx < \delta$$

for all small $t > 0$. The regularity of u increases with ν. As in our accompanying paper on wave-maps [10], the argument is based on a renormalization method for the 'soliton profile' $W(r)$.

1. Introduction

Since the seminal paper of Jörgens [6] much work has been devoted to the study of well-posedness of the nonlinear wave equation

$$\partial_{tt}u - \Delta u + f(u) = 0$$

in $\mathbb{R}^{3+1}_{x,t}$ and suitable nonlinearities $f(u)$. Jörgens showed that for $H^1(\mathbb{R}^3)$ subcritical defocusing nonlinearities $f(u) = |u|^{p-1}u$ with $p < 5$ smooth data lead to smooth solutions for all times. The critical defocusing case $p = 5$ was resolved by Struwe [21] for radial data and Grillakis [5] for general data. These authors proved global well-posedness and scattering results for energy solutions, see Shatah–Struwe [17] and Sogge [18]. No corresponding results are known for the supercritical case $p > 5$.

In this paper we address the solvability of the nonlinear wave equation in $\mathbb{R}^{3+1}$ with a focusing nonlinearity $f(u) = -|u|^{p-1}u$. In this case blow-up may occur. Indeed, it was shown by Levine [11] via a convexity argument that data in $(\dot{H}^1 \cap L^2) \times L^2$ with negative energy lead to finite-time blow-up, see also Strauss [20]. Local well-posedness in the optimal regularity class was considered by several authors, see Sogge [18] for a detailed exposition of this work. Most relevant for us is the case $p = 5$ where the equation is locally well-posed in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^3)$. Moreover, if the solution cannot be continued beyond some finite time T_* as an

The authors were partially supported by the National Science Foundation, J. K. by DMS-0401177, W. S. by DMS-0617854, D. T. by DMS-0354539, and DMS-0301122.

energy solution, then necessarily the Strichartz norm $\|u\|_{L^8([0,T_*)\times\mathbb{R}^3)} = \infty$ (with similar results in all dimensions).

The question of the blow-up rate was addressed by Merle–Zaag in the conformal range $p \le 3$, see [13]–[15] (their results extend to all dimensions). They showed that if solutions to the Cauchy problem, with $1 < p \le 3$,

$$\partial_{tt} u - \Delta u - |u|^{p-1}u = 0, \quad (u_0, u_1) \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$$

blow up in finite time T_*, then the following holds: for any $a \in \mathbb{R}^3$ the self-similar change of variables

$$u(x,t) = (T_* - t)^{-\frac{2}{p-1}} w_a(y,s), \quad y = \frac{x-a}{T_* - t}, \; s = -\log(T_* - t)$$

leads to functions w_a satisfying

$$\sup_{s \ge -\log T_* + 1,\; a \in \mathbb{R}^3} \|w_a(s)\|_{H^1(B)} + \|\partial_s w_a(s)\|_{L^2(B)} \le K$$

where B is the unit ball and a constant K that only depends on p, T_* and the norm of the initial data in $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$.

For the energy critical case $p = 5$, i.e.,

$$\partial_{tt} u - \Delta u - u^5 = 0 \tag{1.1}$$

there has been some recent activity, see [9], [8], [7], which we now describe in more detail. The Talenti–Aubin solutions

$$W(r) = (1 + r^2/3)^{-\frac{1}{2}}$$

are extremizers of the Sobolev imbedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and satisfy the Euler-Lagrange equation $-\Delta W - W^5 = 0$. In [9] the first two authors showed that there exists a small co-dimension one manifold $\mathcal{M}$ around W in a suitable topology so that data on this manifold exhibit global existence and an asymptotic behavior of bulk term plus radiation. The radiation term is also shown to scatter like a free energy wave. It is conjectured, see [2], that this manifold has the property that it separates a region of scattering from one of blow-up. As a first result in this direction, Karageorgis–Strauss [7] showed that above the tangent space of $\mathcal{M}$ at W finite time blow-up occurs, albeit for the equation

$$\partial_{tt} u - \Delta u - |u|^5 = 0$$

Note that the result of [9] equally well applies to the nonlinearity $|u|^5$ (in fact, the solutions constructed in [9] are positive so that there is no distinction between u^5 and $|u|^5$ from the point of view of that paper).

Kenig–Merle [8] studied the behavior of solutions with data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$ of energy $\mathcal{E}(u_0, u_1) < \mathcal{E}(W, 0)$ where the conserved energy is

$$\mathcal{E}(u, u_t) = \int_{\mathbb{R}^3} \left[\frac{1}{2}(u_t^2 + |\nabla u|^2) - \frac{|u|^6}{6}\right] dx$$

They found that in this regime there is a dichotomy between blow-up and global existence/scattering depending on whether $\|\nabla u_0\|_2 > \|\nabla W\|_2$ or $\|\nabla u_0\|_2 < \|\nabla W\|_2$.

Note that

$$W(x, \lambda) := \lambda^{\frac{1}{2}} W(\lambda x)$$

is a stationary solution of (1.1) for all $\lambda > 0$. Moreover, the energy is constant in λ (reflecting the energy criticality of the equation). Linearizing the wave equation around W leads to the linearized operator

$$H = -\Delta - 5W^4$$

The wave evolution of H has two types of instabilities: an exponential instability arising from the negative spectrum of H (which has a unique negative eigenvalue) as well as a "bound state" at zero energy: $H(\partial_\lambda W|_{\lambda=1}) = 0$ where $\partial_\lambda W$ decays like r^{-1} and thus does not belong to $L^2(\mathbb{R}^3)$ — this is what one refers to as a zero energy resonance. In this paper we construct blow-up solutions by 'projecting out' the exponentially growing mode of the linearized equation.

More precisely, we seek radial, real-valued, blow-up solutions

$$u(x,t) = \lambda(t)^{\frac{1}{2}} W(\lambda(t)x) + \eta(x,t)$$

of (1.1) where $\lambda(t) \to \infty$ as $t \to 0$ and with the local energy inside the light-cone $|x| \le t$ of $\eta(x,t)$ going to zero as $t \to 0$. The *local* energy relative to the origin is defined as

$$\mathcal{E}_{\rm loc}(\eta) = \int_{[|x|<t]} (\eta_t^2 + |\nabla\eta|^2 + |\eta|^6)\, dx$$

The following theorem is the main result of this paper. The blow-up occurs at time $t = 0$ when solving backwards in time.

Theorem 1.1. *Let $\nu > \frac{1}{2}$ and $\delta > 0$. Then there exists an energy solution u of* (1.1) *which blows up precisely at $r = t = 0$ and which has the following property: in the cone $|x| = r \le t$ and for small times t the solution has the form, with $\lambda(t) = t^{-1-\nu}$,*

$$u(x,t) = \lambda^{\frac{1}{2}}(t) W(\lambda(t)r) + \eta(x,t)$$

where $\mathcal{E}_{\rm loc}(\eta(\cdot,t)) \to 0$ as $t \to 0$ and outside the cone $u(x,t)$ satisfies

$$\int_{[|x|\ge t]} \big[|\nabla u(x,t)|^2 + |u_t(x,t)|^2 + |u(x,t)|^6\big]\, dx < \delta$$

for all sufficiently small $t > 0$. In particular, the energy of these blow-up solutions can be chosen arbitrarily close to $\mathcal{E}(W,0)$, i.e., the energy of the stationary solution.

The restriction $\nu > 1/2$ arises only due to technical reasons, and we hope to eliminate it in subsequent work. If $\nu > 1$, then the solutions from Theorem 1.1 belong to $L^\infty(\mathbb{R}^3)$ for all small $t > 0$ and blow up at the rate

$$\|u(\cdot,t)\|_\infty \asymp t^{-(1+\nu)/2}$$

as $t \to 0$. The proof is based on a renormalization procedure analogous to the one that the authors used for the construction of blow-up solutions for wave maps in [10]. For our purposes this refers to the fact that we do not simply perturb around $\lambda^{\frac{1}{2}}(t)W(\lambda(t)r)$ to obtain the linearized equation for η, but rather first modify the blow-up profile and then perturb around this "renormalized" profile. More precisely, fix a large integer N. Then there exists a function u^e satisfying

$$u^e \in C^{\frac{\nu+1}{2}-}(\{t_0 > t > 0,\ |x| \le t\}), \qquad \mathcal{E}_{\rm loc}(u^e)(t) \lesssim (t\lambda(t))^{-1} \ \text{ as } \ t \to 0 \tag{1.2}$$

so that the radiation term η above has the form

$$\eta(x,t) = u^e(r,t) + \varepsilon(r,t), \qquad 0 \le r \le t$$

where ε decays at $t = 0$. In fact, η can be extended globally with the property that

$$\varepsilon \in t^N H^{\frac{\nu+2}{2}-}(\mathbb{R}^3), \qquad \varepsilon_t \in t^{N-1} H^{\frac{\nu}{2}-}(\mathbb{R}^3), \qquad \mathcal{E}_{\rm loc}(\varepsilon)(t) \lesssim t^N \ \text{ as } \ t \to 0$$

with spatial norms that are uniformly controlled as $t \to 0$.

As this paper was written concurrently with our wave-map paper [10] it is only natural that there would be some similarities between this paper and [10]. In fact, a secondary goal here is to show that the method used in both papers is flexible and applies to quite distinct scenarios. The main differences between this paper and [10] are as follows:

- The blow-up profile is not constant in L^∞ but rather grows at rate $\lambda^{\frac{1}{2}}$. The renormalization procedure thus needs to be adapted to this case.
- In contrast to [10], the linearized operator exhibits negative spectrum. This produces exponential instability of the linearized wave flow.
- The linearized operator no longer exhibits a strongly singular potential in the sense of [4]. Thus, a (Dirichlet) boundary condition is needed at $R = 0$.

We feel that the most important difference listed here is the exponential instability. In fact, as in the asymptotic stability paper [9], we need to 'project out' this exponential growth. Our blow-up rates are therefore expected to be non-generic.

2. The renormalization step

In this section we show how to construct an arbitrarily good approximate radial solution to the wave equation (1.1) as a perturbation of a time-dependent ground state profile

$$u_0 = \lambda^{\frac{1}{2}} W(R), \quad W(R) = (1 + R^2/3)^{-\frac{1}{2}}, \quad R = r\lambda(t)$$

with the polynomial timescale

$$\lambda(t) = t^{-1-\nu}, \qquad \nu > 0.$$

Theorem 2.1. *Let $k \in \mathbb{N}$. There exists an approximate solution u_{2k-1} for* (1.1) *of the form*

$$u_{2k-1}(r,t) = \lambda^{\frac{1}{2}}(t)\Big[W(R) + \frac{c}{(t\lambda)^2} R^2 (1+R^2)^{-1/2} + O\left(\frac{R^2(1+R^2)^{-\frac{3}{2}}}{(t\lambda)^2}\right)\Big]$$

so that the corresponding error has size

$$e_{2k-1} = O\left(\frac{\lambda^{\frac{1}{2}} R}{t^2 (t\lambda)^{2k}}\right)$$

Here the $O(\cdot)$ terms are uniform in $0 \le r \le t$ and $0 < t < t_0$ where t_0 is a fixed small constant.

Remark 2.2. The u^e in (1.2) is

$$u^e(r,t) = \lambda^{\frac{1}{2}}(t)\Big[\frac{c}{(t\lambda)^2} R^2 (1+R^2)^{-1/2} + O\left(\frac{R^2(1+R^2)^{-\frac{3}{2}}}{(t\lambda)^2}\right)\Big]$$

The analysis below shows that it has the stated regularity up to the light-cone. Moreover, one checks that

$$\mathcal{E}_{\rm loc}\Big(\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2} R\Big) \lesssim (t\lambda)^{-1}$$

which is the claimed decay rate for the local kinetic energy of u^e. The local *potential* energy of u^e decays like $(t\lambda)^{-3}$.

Proof. We iteratively construct a sequence u_k of better approximate solutions by adding corrections v_k,

$$u_k = v_k + u_{k-1}$$

The error at step k is

$$e_k = (-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u_k + u_k^5$$

If u were an exact solution, then the difference

$$\varepsilon = u - u_{k-1}$$

would solve the equation

$$(-\partial_t^2+\partial_r^2+\frac{2}{r}\partial_r)\varepsilon+5u_{k-1}^4\varepsilon+10u_{k-1}^3\varepsilon^2+10u_{k-1}^2\varepsilon^3+5u_{k-1}\varepsilon^4+\varepsilon^5+e_{k-1}=0 \tag{2.1}$$

In a first approximation we linearize this equation around $\varepsilon = 0$ and substitute u_{k-1} by u_0. Then we obtain the linear approximate equation

$$\left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4\right)\varepsilon + e_{k-1} \approx 0 \tag{2.2}$$

For $r \ll t$ we expect the time derivative to play a lesser role so we neglect it and we are left with an elliptic equation with respect to the variable r,

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4\right)\varepsilon + e_{k-1} \approx 0, \qquad r \ll t \tag{2.3}$$

For $r \approx t$ we can approximate u_0^4 by zero and rewrite (2.2) in the form

$$\left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right)\varepsilon + e_{k-1} \approx 0 \tag{2.4}$$

Here the time and spatial derivatives have the same strength. However, we can identify another principal variable, namely $a = r/t$ and think of ε as a function of (t, a). Later, we reduce the above equation to a Sturm-Liouville problem in a which becomes singular at $a = 1$.

The above heuristics lead us to a two step iterative construction of the v_k's. The two steps successively improve the error in the two regions $r \ll t$, respectively $r \approx t$. To be precise, we define v_k by

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4\right)v_{2k+1} + e_{2k}^0 = 0 \tag{2.5}$$

respectively

$$\left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right)v_{2k} + e_{2k-1}^0 = 0 \tag{2.6}$$

both equations having zero Cauchy data[1] at $r = 0$. Here at each stage the error term e_k is split into a principal part and a higher order term (to be made precise below),

$$e_k = e_k^0 + e_k^1$$

[1]The coefficients are singular at $r = 0$, therefore this has to be given a suitable interpretation

The successive errors are then computed as

$$e_{2k} = e^1_{2k-1} + N_{2k}(v_{2k}), \qquad e_{2k+1} = e^1_{2k} - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1})$$

where

$$N_{2k+1}(v) = 5(u_{2k}^4 - u_0^4)\, v + 10u_{2k}^3\, v^2 + 10u_{2k}^2\, v^3 + 5u_{2k}\, v^4 + v^5 \tag{2.7}$$

respectively

$$N_{2k}(v) = 5u_{2k-1}^4 v + 10u_{2k-1}^3 v^2 + 10u_{2k-1}^2 v^3 + 5u_{2k-1} v^4 + v^5 \tag{2.8}$$

To formalize this scheme we need to introduce suitable function spaces in the cone

$$\mathcal{C}_0 = \{(r,t) \,:\, 0 \le r < t,\, 0 < t < t_0\}$$

for the successive corrections and errors. We first consider the a dependence. For the corrections v_k we set

$$\beta_0 = \frac{\nu - 1}{2} > -\frac{1}{2}$$

and use

Definition 2.3. *For $i \in \mathbb{N}$ we let $j(i) = 0$ if ν is irrational, respectively $j(i) = i$ if ν is rational.*

a) For any positive integer k, we define $\mathcal{Q}$ to be the algebra of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:

(i) q is analytic in $[0,1)$ with an even expansion at 0.

(ii) Near $a = 1$ we have an absolutely convergent expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{i(\beta_0+1) - 2\left[\frac{i-1}{4}\right]} \sum_{j=0}^{j(i)} q_{ij}(a)(\log(1-a))^j$$

with analytic coefficients q_0, q_{ij}.

b) $\mathcal{Q}_m$ is the algebra which is defined similarly, with the additional requirement that

$$q_{ij}(1) = 0 \quad \text{if} \quad i = 4k+1 \ge 4m+1.$$

We remark that the exponents of $1-a$ in the above series are all positive because of $\beta_0 > -\frac{1}{2}$. For the errors e_k we introduce

Definition 2.4. *$\mathcal{Q}'$ is the space of continuous functions $q : [0,1) \to \mathbb{R}$ with the following properties:*

(i) q is analytic in $[0,1)$ with an even expansion at 0

(ii) Near $a = 1$ we have a convergent expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{i(\beta_0+1) - 2\left[\frac{i-1}{4}\right] - 1} \sum_{j=0}^{j(i)} q_{ij}(a)(\log(1-a))^j$$

with analytic coefficients q_0, q_{ij}.

b) $\mathcal{Q}'_m$ is the space which is defined similarly, with the additional requirement that

$$q_{ij}(1) = 0 \quad \text{if} \quad i = 4k+1 \ge 4m+1.$$

By construction, $\mathcal{Q}_k \subset \mathcal{Q}'_k$. The families $\mathcal{Q}'$ and $\mathcal{Q}'_k$ are obtained by applying $a^{-1}\partial_a$ to the algebras $\mathcal{Q}$ and $\mathcal{Q}_k$, respectively.

We remark that the number of logarithms in these definitions in the case when ν is rational is far from optimal, but we have chosen this form since it simplifies the presentation. Next we define the class of functions of R:

Definition 2.5. *$S^m(R^k)$ is the class of analytic functions $v : [0,\infty) \to \mathbb{R}$ with the following properties:*

(i) v vanishes of order m and $R^{-m}v$ has an even Taylor expansion at $R = 0$.

(ii) v has a convergent expansion near $R = \infty$,

$$v = \sum_{i=0}^{\infty} c_i \, R^{k-2i}$$

The importance of even expansions in R lies with the fact that only those correspond to smooth functions in $\mathbb{R}^3$. For the same reason, we will work with even m. We also introduce another auxiliary variable,

$$b = \frac{1}{(t\lambda)^2} \tag{2.9}$$

Since we seek solutions inside the cone we can restrict b to a small interval $[0, b_0]$. We combine these three components in order to obtain the full function class which we need:

Definition 2.6. *a) $S^m(R^k, \mathcal{Q}_n)$ is the class of analytic functions $v : [0,\infty) \times [0,1] \times [0, b_0] \to \mathbb{R}$ so that*

(i) v is analytic as a function of R, b,

$$v : [0,\infty) \times [0, b_0] \to \mathcal{Q}_n$$

(ii) v vanishes of order m and $R^{-m}v$ has an even Taylor expansion at $R = 0$.

(iii) v has a convergent expansion at $R = \infty$,

$$v(R, \cdot, b) = \sum_{i=0}^{\infty} c_i(\cdot, b) R^{k-2i}$$

where the coefficients $c_i : [0, b_0] \to \mathcal{Q}_m$ are analytic with respect to b.

b) $\mathrm{IS}^m(R^k, \mathcal{Q}_n)$ is the class of analytic functions w on the cone $\mathcal{C}_0$ which can be represented as

$$w(r,t) = v(R, a, b), \qquad v \in S^m(R^k, \mathcal{Q}_n)$$

We note that the representation of functions on the cone as in part (b) is in general not unique since R, a, b are dependent variables. Later we shall exploit this fact and switch from one representation to another as needed. We shall prove by induction that the successive corrections v_k and the corresponding error terms e_k can be chosen with the following properties: For each $k \geq 1$,

$$v_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^2(R, \mathcal{Q}_{k-1}) \tag{2.10}$$

$$t^2 e_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^0(R, \mathcal{Q}'_{k-1}) \tag{2.11}$$

$$v_{2k} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k+2}} \, \mathrm{IS}^2(R^3, \mathcal{Q}_k) \tag{2.12}$$

$$t^2 e_{2k} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \left[\mathrm{IS}^0(R^{-1}, \mathcal{Q}_k) + b \, \mathrm{IS}^0(R, \mathcal{Q}'_k) \right] \tag{2.13}$$

with (2.13) also valid for $k = 0$. We remark that the order of vanishing at $R = 0$ can be successively improved with k, but this does not appear to be important.

Step 0: *The analysis at $k = 0$*

With our notations, one checks that

$$t^2 e_0 = -t^2 \partial_{tt} [\lambda^{\frac{1}{2}} W(\lambda(t) r)] \in \lambda^{\frac{1}{2}} IS^0(R^{-1}) \tag{2.14}$$

as claimed. Now assume we know the above relations hold up to $k-1$ with $k \geq 1$, and we show how to construct v_{2k-1}, respectively v_{2k}, so that they hold for the index k.

Step 1: *Begin with e_{2k-2} satisfying* (2.13) *or* (2.14) *and choose v_{2k-1} so that* (2.10) *holds.*

If $k = 1$, then define $e_0^0 := e_0$. If $k > 1$, we use (2.13) to write

$$e_{2k-2} = e^0_{2k-2} + e^1_{2k-2}$$

where

$$t^2 e^0_{2k-2} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-2}} \, \mathrm{IS}^0(R^{-1}, \mathcal{Q}_{k-1}), \qquad t^2 e^1_{2k-2} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^0(R, \mathcal{Q}'_{k-1})$$

In the first term we can set $b = 0$ and eliminate the b dependence, as all the b dependent part can be included in the second term.

We note that the term e^1_{2k-2} can be included in e_{2k-1}, cf. (2.11). We define v_{2k-1} as in (2.5) neglecting the a dependence of e^0_{2k-2}. In other words, a is treated as a parameter. Changing variables to R in (2.5) we need to solve the equation

$$(t\lambda)^2 L v_{2k-1} = t^2 e^0_{2k-2} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-2}} \, \mathrm{IS}^0(R^{-1}, \mathcal{Q}_{k-1})$$

where the operator L is given by

$$L = -\partial_R^2 - \frac{2}{R} \partial_R - 5W^4(R)$$

Then (2.10) is a consequence of the following ODE lemma.

Lemma 2.7. *The solution v to the equation*

$$Lv = f \in S^0(R^{-1}), \qquad v(0) = v'(0) = 0$$

has the regularity

$$v \in S^2(R)$$

Proof. Since f is analytic at 0 with a constant leading term, one can easily write down an even Taylor series for v at 0 with a quadratic leading term.

It remains to determine the asymptotic behavior of v at infinity. For this it is convenient to remove the first order derivative in L (to achieve constancy of the Wronskian). Thus, we seek a solution of

$$\tilde{L} R\, v = Rf, \qquad \tilde{L} = \partial_R^2 + 5W^4 = \partial_R^2 + \frac{5}{(1+R^2/3)^2}$$

We use this fundamental system of solutions for $\tilde{L}$:

$$\phi(R) = R(1-R^2/3)(1+R^2/3)^{-\frac{3}{2}}$$
$$\theta(R) = (1+R^2/3)^{-\frac{3}{2}}(1-2R^2+R^4/9)$$

Clearly, $L\partial_\lambda W = 0$ and we set $\phi = R\partial_\lambda W|_{\lambda=1}$ up to a constant. The function θ is then determined from the Wronskian constancy condition $W(\theta,\phi) = 1$. This allows us to obtain an integral representation for v using the variation of parameters formula, which gives

$$v = -R^{-1}\theta(R)\int_0^R \phi(R')R'f(R')\ dR' + R^{-1}\phi(R)\int_0^R \theta(R')R'f(R')\ dR'$$

The right-hand side grows like R, as claimed. $\square$

As a special case of the above computation we note the representation for v_1,

$$v_1 = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2}V(R), \qquad V \in S^2(R) \tag{2.15}$$

This justifies the choice of the second term in the expansion for u_{2k-1} in Theorem 2.1.

Step 2: *Show that if* v_{2k-1} *is chosen as above then* (2.11) *holds.*

Thinking of v_{2k-1} as a function of t, R and a we can write e_{2k-1} in the form

$$e_{2k-1} = N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

Here $N_{2k-1}(v_{2k-1})$ accounts for the contribution from the nonlinearity and is given by (2.7). $E^t v_{2k-1}$ contains the terms in

$$-\partial_{tt} v_{2k-1}(t,R,a) \tag{2.16}$$

where no derivative applies to the variable a, while $E^a v_{2k-1}$ contains those terms in

$$\Big(\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r\Big) v_{2k-1}(t,R,a)$$

where at least one derivative applies to the variable a (recall that in Step 1 the parameter a was frozen). We begin with the terms in N_{2k-1}. We first note that, by summing the v_j over $1 \le j \le 2k-2$,

$$u_{2k-2} - u_0 \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2}\, \mathrm{IS}^2(R, \mathcal{Q}_{k-1}) \tag{2.17}$$

The first term in $N_{2k-1}(v_{2k-1})$ contributes

$$\begin{aligned} t^2(u_{2k-2}^4 - u_0^4)v_{2k-1} \;&=\; t^2[(u_{2k-2}-u_0)^4 + 4(u_{2k-2}-u_0)^3 u_0 \\ &\quad +6(u_{2k-2}-u_0)^2u_0^2 + 4(u_{2k-2}-u_0)u_0^3]v_{2k-1} \end{aligned} \tag{2.18}$$

Using (2.17) we compute

$$
\begin{aligned}
t^2(u_{2k-2}-u_0)^4 v_{2k-1} &\in \frac{1}{(t\lambda)^6}\,\mathrm{IS}^8(R^4,\mathcal{Q}_{k-1})\frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R,\mathcal{Q}_{k-1})\\
&\subset a^6\,\mathrm{IS}^2(R^{-2},\mathcal{Q}_{k-1})\frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R,\mathcal{Q}_{k-1})\\
&\subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1})
\end{aligned}
$$

as well as

$$
\begin{aligned}
t^2(u_{2k-2}-u_0)u_0^3 v_{2k-1} &\in t^2\frac{\lambda^{\frac12}}{(t\lambda)^2}\,\mathrm{IS}^2(R,\mathcal{Q}_{k-1})\lambda^{\frac32}S^0(R^{-3})\frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R,\mathcal{Q}_{k-1})\\
&\subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1})
\end{aligned}
$$

The other two terms in (2.18) are similar. Next, compute

$$
\begin{aligned}
t^2 v_{2k-1}^5 &\in \frac{t^2\lambda^{\frac52}}{(t\lambda)^{10k}}\,\mathrm{IS}^{10}(R^5,\mathcal{Q}_{k-1})\\
&\subset \frac{\lambda^{\frac12}R^6}{(t\lambda)^{10k-2}}\,\mathrm{IS}^4(R^{-1},\mathcal{Q}_{k-1})\\
&\subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}a^6 b^{4(k-1)}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1}) \subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1})
\end{aligned}
$$

and

$$
\begin{aligned}
t^2u_{2k-2}^3\,v_{2k-1}^2 &\in \lambda^{-\frac12}(t\lambda)^2\,\mathrm{IS}^0(R^{-3},\mathcal{Q}_{k-1})\frac{\lambda}{(t\lambda)^{4k}}\,\mathrm{IS}^4(R^2,\mathcal{Q}_{k-1})\\
&\subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}b^{2k-2}\,\mathrm{IS}^4(R^{-1},\mathcal{Q}_{k-1}) \subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1})
\end{aligned}
$$

with similar statements for $u_{2k-2}^2v_{2k-1}^3$ and $u_{2k-2}v_{2k-1}^4$. Summing up we obtain

$$
N_{2k-1}(v_{2k-1}) \in \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}_{k-1}) \subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R^{-1},\mathcal{Q}'_{k-1})
$$

This concludes the analysis of $N_{2k-1}(v_{2k-1})$. We continue with the terms in $E^t v_{2k-1}$, where we can neglect the a dependence. Therefore, it suffices to compute

$$
t^2\partial_t^2\left(\frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R)\right) \subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\,\mathrm{IS}^2(R)
$$

Finally, we consider the terms in $E^a v_{2k-1}$. With

$$
v_{2k-1}(r,t) = \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}w(R,a), \quad w\in S^2(R,\mathcal{Q}_{k-1})
$$

we have

$$
\begin{aligned}
t^2E^a v_{2k-1} \quad = \quad & -2t\partial_t\left(\frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\right)aw_a(R,a) + \frac{\lambda^{\frac12}}{(t\lambda)^{2k}}\big[2(\nu+1)aRw_{aR}(R,a)\\
& -2Ra^{-1}w_{Ra} - 2a^{-1}w_a(R,a) + (a^2-1)w_{aa}(R,a) + 2aw_a(R,a)\big]
\end{aligned}
$$

Since $\mathcal{Q}_{k-1}$ are even in a we conclude that

$$(1-a^2)\partial_{aa},\, a\partial_a,\, a^{-1}\partial_a \,:\, \mathcal{Q}_{k-1} \to \mathcal{Q}'_{k-1}$$

and therefore

$$t^2 E^a v_{2k-1} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^2(R, \mathcal{Q}'_{k-1})$$

This concludes the proof of (2.11). We remark that for the special case of $k=1$, i.e., with v_1 as in (2.15), these arguments yield

$$t^2 e_1 \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2} IS^2(R) \tag{2.19}$$

Step 3: *Define v_{2k} so that* (2.12) *holds.*

We begin the analysis with e_{2k-1} replaced by its main asymptotic component $\tilde{e}^0_{2k-1}$ around $R=\infty$. This has the form

$$t^2 \tilde{e}^0_{2k-1} = \frac{\lambda^{\frac{1}{2}} R}{(t\lambda)^{2k}} q(a), \qquad q \in \mathcal{Q}'_{k-1} \tag{2.20}$$

which we rewrite as

$$t^2 \tilde{e}^0_{2k-1} = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} a q(a)$$

We remark that (2.19) implies that

$$t^2 \tilde{e}^0_1(a) = \frac{\lambda^{\frac{1}{2}}}{t\lambda} a.$$

Consider the equation (2.6) with $\tilde{e}^0_{2k-1}$ on the right-hand side,

$$t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) \tilde{v}_{2k} = -t^2 \tilde{e}^0_{2k-1}$$

We look for a solution $\tilde{v}_{2k}$ which has the form

$$\tilde{v}_{2k} = -\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} W_{2k}(a)$$

Thus,

$$t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) \left(\frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} W_{2k}(a)\right) = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} a q(a)$$

Conjugating out the power of t we get

$$t^2 \Big(- \Big(\partial_t + \frac{(2k-\frac{3}{2})\nu - \frac{1}{2}}{t}\Big)^2 + \partial_r^2 + \frac{2}{r}\partial_r\Big) W_{2k}(a) = a q(a)$$

which we rewrite as an equation in the a variable,

$$L_{(2k-\frac{3}{2})\nu-\frac{1}{2}}\, W_{2k}(a) = a q(a), \qquad q \in \mathcal{Q}'_{k-1} \tag{2.21}$$

where the one-parameter family of operators L_β is defined by

$$L_\beta = (1-a^2)\partial_{aa} + 2(a^{-1} + a\beta - a)\partial_a - \beta^2 + \beta \tag{2.22}$$

We claim that solving this equation with zero Cauchy data at $a=0$ yields a solution which satisfies

$$W_{2k}(a) = a^3 q_1(a), \qquad q_1 \in \mathcal{Q}_k \tag{2.23}$$

This gives

$$\tilde{v}_{2k} = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k-1}} a^3 q_1(a) = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k+2}} R^3 q_1(a)$$

which is not entirely suitable as the next correction since it has an odd expansion at $R = 0$ instead of an even one. To remedy this we simply set

$$v_{2k} = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k+2}} R^3 \frac{R}{(1+R^2)^{\frac{1}{2}}} q_1(a) \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k+2}} S^4(R^3, \mathcal{Q}_k)$$

Clearly, this will conclude Step 3. To prove the claim (2.23) we need the following

Lemma 2.8. *Let $f \in a\mathcal{Q}'_{k-1}$, $k \geq 1$. Then there is a unique solution $w \in a^3\mathcal{Q}_k$ to the equation*

$$L_{(2k-\frac{3}{2})\nu-\frac{1}{2}}\, w = f, \qquad w(0) = 0,\ \partial_a w(0) = 0 \tag{2.24}$$

Proof. Denote

$$\beta = \big(2k - \frac{3}{2}\big)\nu - \frac{1}{2} = (4k-3)\beta_0 + 2(k-1) > -\frac{1}{2}. \tag{2.25}$$

We write

$$L_\beta = a^{-2}\partial_a(a^2\partial_a) - a^2\partial_{aa} + 2(\beta-1)a\partial_a - \beta^2 + \beta$$

To study the behavior of the solutions at 0 we match the coefficients in $L_\beta\, w = f$ with

$$f(a) = \sum_{j=1}^\infty f_j\, a^{2j-1}, \qquad w(a) = \sum_{j=2}^\infty w_j\, a^{2j-1}$$

yields the system, with $j \geq 1$,

$$(2j+1)(2j+2)w_{j+1} + [-(2j-1)(2j-2) + 2(\beta-1)(2j-1) + (\beta-\beta^2)]w_j = f_j$$

where we take $w_1 = 0$. The coefficient of w_{j+1} is always nonzero; this allows us to successively compute the coefficients w_j. The convergence of the series for w follows from the convergence of the series for f.

It remains to study the solution w near $a = 1$. The behavior of L_β at 1 is well approximated by

$$L^1_\beta = 2(1-a)\partial_{aa} + 2\beta\partial_a = 2(1-a)^{\beta+1}\partial_a[(1-a)^{-\beta}\partial_a]$$

which annihilates the functions 1 and $(1-a)^{\beta+1}$. Therefore, we seek a fundamental system for $L_\beta y = 0$ of the form

$$\phi_1(a) = 1 + \sum_{\ell=1}^\infty \mu_\ell (1-a)^\ell, \qquad \phi_2(a) = (1-a)^{\beta+1}\Big[1 + \sum_{\ell=1}^\infty \tilde{\mu}_\ell (1-a)^\ell\Big] \tag{2.26}$$

This leads to the conditions, with $\mu_0 = \tilde{\mu}_0 = 1$,

$$2\mu_{\ell+1}(\ell+1)(\ell-\beta) = \mu_\ell[\ell(\ell-2\beta+3) + \beta^2 - \beta] + 2\sum_{j=0}^{\ell-1} j\mu_j \tag{2.27}$$

$$\begin{aligned} 2\tilde{\mu}_{\ell+1}(\ell+\beta+2)(\ell+1) &= \tilde{\mu}_\ell[(\beta+1+\ell)(\ell-\beta+4) + \beta^2 - \beta] \\ &\quad + 2\sum_{j=0}^{\ell-1}(\beta+1+j)\tilde{\mu}_j \end{aligned} \tag{2.28}$$

Clearly, (2.28) always has a solution whereas (2.27) requires $\beta \notin \mathbb{Z}_0^+$; in the latter case, the series in (2.26) define entire functions. If, on the other hand, $\beta \in \mathbb{Z}_0^+$, then ϕ_1 is modified to

$$\phi_1(a) = 1 + \sum_{\ell=1}^{\infty} \mu_\ell (1-a)^\ell + c_1 \phi_2(a) \log(1-a) \tag{2.29}$$

with some unique choice of c_1.

Modulo a linear combination of ϕ_1, ϕ_2 it suffices to find one solution to the inhomogeneous equation $L_\beta\, w = f$ near $a = 1$. We begin with the case when f has the form

$$f(a) = (1-a)^\gamma \sum_{k=0}^{\infty} f_k (1-a)^k$$

with $\gamma > 0$. Then we seek a solution w of the form

$$w(a) = (1-a)^{\gamma+1} \sum_{k=0}^{\infty} w_k (1-a)^k$$

This leads to the system

(2.30)
$$\begin{cases} 2w_0(\gamma+1)(\gamma-\beta) & = f_0 \\ 2w_{\ell+1}(\gamma+\ell+2)(\gamma-\beta+\ell+1) & = f_\ell + w_\ell[(\gamma+\ell+1)(\gamma+\ell+4-2\beta)+\beta^2-\beta] \\ & \quad +2\sum_{j=0}^{\ell-1}(\gamma+j+1)w_j \end{cases}$$

which is solvable unless $\beta - \gamma$ is a nonnegative integer, in which case the solution w has the modified form

$$w(a) = (1-a)^{\gamma+1} \sum_{k=0}^{\infty} w_k (1-a)^k + c\phi_2(a)\log(1-a)$$

Indeed, assume that $\beta - \gamma = p \geq 0$ with integer p. Then (2.30) is in general violated for $\ell + 1 = p$ regardless of the choice of w_p; thus, c above has to be chosen so that the coefficients of $(1-a)^{\gamma+p} = (1-a)^\beta$ in $L_\beta w = f$ match. This can be done since

$$\phi_2(a)\log(1-a) = [(1-a)^{\beta+1} + O((1-a)^{\beta+2})]\log(1-a)$$

and

$$L_\beta[\phi_2(a)\log(1-a)] = 2(\beta+1)(1-a)^\beta + O((1-a)^{\beta+1})$$

with $\beta + 1 > \frac{1}{2}$ and thus nonzero. Note that w_p remains undetermined – this amounts to the freedom of adding a constant multiple of $\phi_2(a)$ to $w(a)$.

Similarly, if f has the form

$$f(a) = \sum_{m=0}^{j} (\log(1-a))^m f_m(a) = \sum_{m=0}^{j} (\log(1-a))^m (1-a)^\gamma \sum_{k=0}^{\infty} f_{km}(1-a)^k$$

then we seek a solution w of the form

$$w(a) = \sum_{m=0}^{j} (\log(1-a))^m w_m(a) = \sum_{m=0}^{j} (\log(1-a))^m (1-a)^{\gamma+1} \sum_{k=0}^{\infty} w_{km}(1-a)^k$$

Identifying the coefficients of the powers of $\log(1-a)$ we obtain the system

$$L_\beta w_j = f_j, \qquad L_\beta w_{j-1} = f_{j-1} - jQ_1 w_j,$$

respectively

$$L_\beta w_{j-k} = f_{j-1} - jQ_1 w_{j-k+1} + j(j-1)Q_2 w_{j-k+2}, \qquad k \geq 2$$

where

$$Q_1 = 2(1+a)\partial_a + \frac{1+a}{1-a} + \frac{2(1+a)}{a} + \frac{2a\beta}{1-a}, \qquad Q_2 = \frac{1+a}{1-a}$$

This system is solved iteratively as in the first case provided that $\beta - \gamma$ is not a nonnegative integer. Otherwise, the solution w has the modified form

$$w(a) = \sum_{m=0}^{j} (\log(1-a))^m \big[c_m\, \phi_2(a)\log(1-a) + (1-a)^{\gamma+1} \sum_{k=0}^{\infty} w_{km}(1-a)^k\big]$$

Consider now $f \in \mathcal{Q}'_{k-1}$. If ν is irrational then according to Definition 2.4 we can represent it in the form

$$f(a) = f_0(a) + \sum_{i=1}^{\infty} (1-a)^{i(\beta_0+1)-2\left[\frac{i-1}{4}\right]-1} f_i(a)$$

with f_i analytic. Hence the exponent γ above takes the values

$$\gamma_i = i(\beta_0+1) - 2\left[\frac{i-1}{4}\right] - 1$$

On the other hand, we have

$$\beta = (2k - \frac{3}{2})\nu - \frac{1}{2} = (4k-3)(\beta_0+1) - 2k + 1 = \gamma_{4k-3}$$

Then $\gamma_i - \beta$ can only be an integer if $i = 4k-3$. However, in this case there is no logarithmic term due to the additional condition $q_{4k-3}(1) = 0$ which has the effect of replacing γ_{4k-3} by $\gamma_{4k-3}+1$. The conclusion of the lemma follows in the irrational case.

On the other hand, if ν is rational then f has the representation

$$f(a) = f_0(a) + \sum_{i=1}^{\infty} (1-a)^{i(\beta_0+1)-2\left[\frac{i-1}{4}\right]-1} \sum_{j=0}^{i} f_{ij}(a)(\log(1-a))^j$$

with f_{ij} analytic. In order for $\beta - \gamma_\ell$ to be a nonnegative integer we need to have $\ell \leq 4k-3$. Again the condition $q_{4k-3}(1) = 0$ guarantees that there is no extra logarithmic term arrising from the $i = 4k-3$ component of f. On the other hand, if $\ell < 4k-3$ then we can contribute an extra logarithm for a total of $\ell + 1 \leq 4k-3$ logarithms to the $i = 4k-3$ term of w. The conclusion of the lemma again follows.

It is also worth discussing the special case $k = 1$. This will also serve to explain how the algebra $\mathcal{Q}_k$ arises in the iteration. If $k = 1$, then (2.21) reduces to the equation, with the usual $\beta_0 = (\nu-1)/2$,

$$L_{\beta_0} W_2(a) = a$$

due to $t^2 f_1(a) = \lambda^{\frac{1}{2}}(t\lambda)^{-1}a$. As discussed above,

$$W_2(a) = g_0(a) + g_1(a)(1-a)^{\beta_0+1} \quad \text{if } \beta_0 \notin \mathbb{Z}_0^+$$
$$W_2(a) = h_0(a) + h_1(a)(1-a)^{\beta_0+1} + h_2(a)(1-a)^{\beta_0+1}\log(1-a) \quad \text{if } \beta_0 \in \mathbb{Z}_0^+$$

Thus, we see that in all cases $W_2 \in \mathcal{Q}_1$ for $j = 0, 1$ and a near 1. □

Step 4: *With v_{2k} as above show that e_{2k} is as claimed.*

Modifying (2.20) to insure an even expansion at $R = 0$ we set

$$t^2 e^0_{2k-1} = \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} R \frac{R}{(R^2+1)^{\frac{1}{2}}} q(a), \qquad q \in \mathcal{Q}'_{k-1}$$

Then we can write e_{2k} in the form

$$t^2 e_{2k} = t^2 \left(e_{2k-1} - e^0_{2k-1}\right) + t^2 \left(e^0_{2k-1} - \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) v_{2k}\right) + t^2 N_{2k}(v_{2k})$$

where N_{2k} is defined by (2.8).

We begin with the first term in e_{2k}, which has the form

$$t^2(e_{2k-1} - e^0_{2k-1}) \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^0(R^{-1}, \mathcal{Q}'_{k-1})$$

We claim that

$$\mathrm{IS}^0(R^{-1}, \mathcal{Q}'_{k-1}) \subset \mathrm{IS}^0(R^{-1}, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^2} \, \mathrm{IS}^2(R, \mathcal{Q}'_{k-1}) \tag{2.31}$$

For $w \in \mathrm{IS}^1(R^{-1}, \mathcal{Q}'_{k-1})$ we write

$$w = (1-a^2)w + \frac{1}{(t\lambda)^2} R^2 w$$

Then

$$(1-a^2)w \in \mathrm{IS}^0(R^{-1}, \mathcal{Q}_{k-1}), \qquad \frac{1}{(t\lambda)^2} R^2 w \in \frac{1}{(t\lambda)^2} \, \mathrm{IS}^2(R, \mathcal{Q}'_{k-1})$$

as desired.

Next, we consider the expression

$$f = t^2 \left(e^0_{2k-1} - \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) v_{2k}\right)$$

By construction this would vanish if the factor $R(1+R^2)^{-\frac{1}{2}}$ were dropped in both e^0_{2k-1} and v_{2k}. Hence f contains only the components of the second term for which at least one derivative falls on the factor $R(1+R^2)^{-\frac{1}{2}}$:

$$f = \tilde{v}_{2k} t^2 \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) \frac{R}{(1+R^2)^{\frac{1}{2}}} - 2t^2 \partial_t \tilde{v}_{2k} \partial_t \frac{R}{(1+R^2)^{\frac{1}{2}}}$$
$$+ 2t^2 \partial_r \tilde{v}_{2k} \partial_r \frac{R}{(1+R^2)^{\frac{1}{2}}} + \tilde{v}_{2k} \frac{2t^2}{r} \partial_r \frac{R}{(1+R^2)^{\frac{1}{2}}}$$

where $v_{2k} = \tilde{v}_{2k} R(1+R^2)^{-\frac{1}{2}}$. Then a direct computation gives

$$f \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k+2}} IS^2(R, \mathcal{Q}'_k)$$

as needed.

Finally we consider the nonlinear terms in $N_{2k}(v_{2k})$. Again the a, b dependence is uninteresting since $\mathcal{Q}_k$ is an algebra. We start with the last term in (2.8). By construction,

$$v_{2k} \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^{2k}} \, \mathrm{IS}^2(R, \mathcal{Q}_k)$$

Thus,

$$t^2 v_{2k}^5 \in \frac{t^2 \lambda^{\frac52}}{(t\lambda)^{10k}} \,\mathrm{IS}^5(R^5, \mathcal{Q}_k) \subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}} \frac{1}{(t\lambda)^{8k-2}} (1+R^2)^3 \,\mathrm{IS}^1(R^{-1}, \mathcal{Q}_k)$$

Using that

$$\frac{(1+R^2)^3}{(t\lambda)^{8k-2}} = b^{4k-1} + 3a^2 b^{4k-2} + 3a^4 b^{4k-3} + a^6 b^{4k-4}$$

we conclude that

$$t^2 v_{2k}^5 \in \frac{\lambda^{\frac12}}{(t\lambda)^{2k}} \,\mathrm{IS}^1(R^{-1}, \mathcal{Q}_k)$$

which is an admissible contribution to (2.13). Finally, we check the first term in (2.8). From

$$u_{2k-1} - u_0 \in \frac{\lambda^{\frac12}}{(t\lambda)^2} \,\mathrm{IS}^2(R, \mathcal{Q}_k)$$

and the form of v_{2k},

$$\begin{aligned} t^2 u_{2k-1}^4 v_{2k} &\in t^2 \Big(\lambda^{\frac12} W(R) + \frac{\lambda^{\frac12}}{(t\lambda)^2} \,\mathrm{IS}^2(R, \mathcal{Q}_k) \Big)^4 \frac{\lambda^{\frac12}}{(t\lambda)^{2k+2}} \,\mathrm{IS}^2(R^3, \mathcal{Q}_k) \\ &\subset t^2 \Big(\lambda^{\frac12} S^0(R^{-1}) + \lambda^{\frac12} a^2 \,\mathrm{IS}^0(R^{-1}, \mathcal{Q}_k) \Big)^4 \frac{\lambda^{\frac12}}{(t\lambda)^{2k+2}} \,\mathrm{IS}^2(R^3, \mathcal{Q}_k) \\ &\subset (t\lambda)^2 \,\mathrm{IS}^0(R^{-4}, \mathcal{Q}_k) \frac{\lambda^{\frac12}}{(t\lambda)^{2k+2}} \,\mathrm{IS}^2(R^3, \mathcal{Q}_k) \\ &\subset \frac{\lambda^{\frac12}}{(t\lambda)^{2k}} \,\mathrm{IS}^2(R^{-1}, \mathcal{Q}_k) \end{aligned}$$

The other three terms in (2.8) can be checked similarly. This concludes the proof of Theorem 2.1. $\square$

3. The linearized problem

We seek a radial solution of (1.1) of the form $u = u_{2k-1} + \varepsilon$ with u_{2k-1} as in Theorem 2.1. Our ansatz leads to

$$\partial_{tt}\varepsilon - \Delta\varepsilon - 5\lambda^2(t) W^4(\lambda(t)x)\varepsilon = N_{2k-1}(\varepsilon) + e_{2k-1} \tag{3.1}$$

where $N_{2k-1}(\varepsilon)$ is as in (2.7) but with u_{2k-2} replaced with u_{2k-1}. Set $\varepsilon(t,x) = v(\tau(t), \lambda(t)x)$ and $y = \lambda(t)x$, $\tau = \tau(t)$. Then, with $\dot\lambda = \frac{d\lambda}{d\tau}$,

$$\partial_t \varepsilon(t,r) = \tau'(t)(v_\tau + \dot\lambda\lambda^{-1} y \partial_y v)$$

and

$$\partial_{tt}\varepsilon(t,r) = \tau''(t)(\partial_\tau + \dot\lambda\lambda^{-1} y\partial_y)v + \tau'(t)^2 (\partial_\tau + \dot\lambda\lambda^{-1} y \partial_y)^2 v \tag{3.2}$$

Set

$$\tau(t) = \int_t^{t_0} \lambda(s)\, ds + \frac{1}{\nu} t_0^{-\nu} = \frac{1}{\nu} t^{-\nu}$$

so that $\tau'(t) = \lambda(t)$, and $\tau''(t) = \dot\lambda(\tau)\lambda(\tau)$ (we are writing $\lambda(\tau)$ instead of $\lambda(t(\tau))$). Then (3.1) can be rewritten as

$$\begin{aligned} &[(\partial_\tau + \dot\lambda\lambda^{-1} y\partial_y)^2 v + \dot\lambda\lambda^{-1}(\partial_\tau + \dot\lambda\lambda^{-1} y\partial_y)v - \Delta v - 5W^4 v](\tau, y) \\ &= \lambda^{-2}(\tau)[N_{2k-1}(\varepsilon) + e_{2k-1}](t(\tau), \lambda^{-1} y) \end{aligned} \tag{3.3}$$

We remark that the linear Schrödinger operator $H = -\Delta - 5W^4$ on $L^2(\mathbb{R}^3)$ has at least one negative eigenvalue as well as a zero energy eigenvalue and resonance. The negative spectrum renders the linear evolution in (3.3) exponentially unstable. To address this further, we switch to the radial variable $R = \lambda r$ from the previous section. Thus, (3.3) is the same as

$$\begin{aligned}&[(\partial_\tau + \dot\lambda\lambda^{-1}R\partial_R)^2 v + \dot\lambda\lambda^{-1}(\partial_\tau + \dot\lambda\lambda^{-1}R\partial_R)v - v_{RR} - \frac{2}{R}v_R - 5W^4 v](\tau, R)\\ &= \lambda^{-2}(\tau)[N_{2k-1}(\varepsilon) + e_{2k-1}](t(\tau), \lambda^{-1}R)\end{aligned}$$

or, with the new dependent variable $\tilde\varepsilon(\tau, R) := Rv(\tau, R)$,

$$\begin{aligned}&(\partial_\tau + \dot\lambda\lambda^{-1}R\partial_R)^2\tilde\varepsilon - \dot\lambda\lambda^{-1}(\partial_\tau + \dot\lambda\lambda^{-1}R\partial_R)\tilde\varepsilon + \mathcal{L}\tilde\varepsilon\\ &= \lambda^{-2}R[N_{2k-1}(R^{-1}\tilde\varepsilon) + e_{2k-1}]\end{aligned} \tag{3.4}$$

where

$$\mathcal{L} = -\partial_{RR} - 5W^4(R) \quad \text{on} \quad L^2(0,\infty)$$

with a Dirichlet boundary condition at $R = 0$. Let

$$\beta(\tau) := \dot\lambda\lambda^{-1}(\tau) = \frac{1+\nu}{\tau\nu}, \quad \mathcal{D} := \partial_\tau + \beta(\tau)R\partial_R$$

and rewrite (3.4) in the form

$$\mathcal{D}^2\tilde\varepsilon - \beta(\tau)\mathcal{D}\tilde\varepsilon + \mathcal{L}\tilde\varepsilon = f \tag{3.5}$$

To solve this equation we need precise spectral information on the operator $\mathcal{L}$.

4. The spectral and scattering theory of the linearized operator

Definition 4.1. *Let*

$$\mathcal{L} := -\partial_{RR} - \frac{5}{(1 + R^2/3)^2}$$

be the half-line operator on $L^2(0,\infty)$ with a Dirichlet condition at $R = 0$. It is self-adjoint on the domain

$$\mathrm{Dom}(\mathcal{L}) = \{f \in L^2((0,\infty)) \;:\; f, f' \in AC([0,R])\,\forall\, R,\; f(0) = 0,\; f'' \in L^2((0,\infty))\}$$

Note that $\mathcal{L}\phi = 0$ where

$$\phi(R) := 2R\partial_\lambda\Big|_{\lambda=1}\lambda^{\frac12}W(\lambda R) = R(1 - R^2/3)(1 + R^2/3)^{-\frac32}$$

This means that $\mathcal{L}$ has a resonance at zero energy. Since ϕ has a single positive zero, it follows from oscillation theory, see [3], that there is an unique simple negative eigenvalue which we denote by ξ_d. Thus, there is $\phi_d \in L^2(0,\infty) \cap C^\infty([0,\infty))$, decaying exponentially, and with $\phi_d(R) > 0$ for $R > 0$ but $\phi_d(0) = 0$ so that $\mathcal{L}\phi_d = \xi_d\phi_d$. We also assume that $\|\phi_d\|_2 = 1$. Clearly, $\mathcal{L}$ has no other eigenvalues or resonances.

Lemma 4.2. *The spectrum of $\mathcal{L}$ equals*

$$\mathrm{spec}(\mathcal{L}) = \{\xi_d\} \cup [0,\infty).$$

The positive spectrum is purely absolutely continuous, and ξ_d is a simple eigenvalue with eigenfunction that we denote by ϕ_d. Moreover, $\mathcal{L}$ has a resonance at zero. In fact, $\mathcal{L}\phi_0 = 0$ with $\phi_0(R) = R(1 - R^2/3)(1 + R^2/3)^{-\frac32}$. Finally, $\mathcal{L}$ is in the limit-point case at infinity.

As usual, see Marchenko [12] or Section 2 of [4], we introduce the standard fundamental system of solutions $\phi(R,z)$ and $\theta(R,z)$ for all $z\in\mathbb{C}$ of $\mathcal{L}y = zy$ with the boundary conditions

$$\phi(0,z)=\theta'(0,z)=0, \quad \phi'(0,z)=\theta(0,z)=1$$

so that in particular

$$W(\theta(\cdot,z),\phi(\cdot,z))=1$$

These functions are entire in z. Note that $\phi(R,0)=\phi_0(R)$ from above. Furthermore,

$$\theta_0(R):=\theta(R,0)=(1-2R^2+R^4/9)(1+R^2/3)^{-\frac{3}{2}}$$

The Weyl-Titchmarsh function $m(z)$ is uniquely defined by

$$\psi_+(\cdot,z):=\theta(\cdot,z)+m(z)\phi(\cdot,z)\in L^2(0,\infty)\ \forall\ \operatorname{Im} z>0 \tag{4.1}$$

The solution ψ_+ is referred to as the Weyl-Titchmarsh solution. Then one has the following, see [4]:

Proposition 4.3. *The function m can be analytically continued to $\mathbb{C}\setminus\operatorname{spec}(\mathcal{L})$ and it is a Herglotz function. For each $R\geq 0$, $\psi_+(R,z)$ and $\psi_+'(R,z)$ are analytic on $\mathbb{C}\setminus\operatorname{spec}(\mathcal{L})$. The spectral measure of $\mathcal{L}$ is*

$$d\rho=\delta_{\xi_d}+\rho(\xi)d\xi, \qquad \rho(\xi)=\frac{1}{\pi}\operatorname{Im} m(\xi+i0)$$

in the following sense: the distorted Fourier transform defined as

$$\mathcal{F}: f\longrightarrow \hat{f}$$

$$\hat{f}(\xi_d)=\int_0^\infty \phi_d(R)f(R)\,dR, \qquad \hat{f}(\xi)=\lim_{b\to\infty}\int_0^b \phi(R,\xi)f(R)\,dR, \quad \xi\geq 0$$

is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\{\xi_d\}\cup\mathbb{R}^+,\rho)$ and its inverse is given by

$$\mathcal{F}^{-1}:\hat{f}\longrightarrow f(R)=\hat{f}(\xi_d)\phi_d(R)+\lim_{\mu\to\infty}\int_0^\mu \phi(R,\xi)\hat{f}(\xi)\,\rho(\xi)\,d\xi$$

Here lim *refers to the $L^2(\mathbb{R}^+,\rho)$, respectively the $L^2(\mathbb{R}^+)$, limit.*

In the sequel we view the Fourier transform as a vector-valued map

$$f\mapsto \begin{pmatrix}\hat{f}(\xi_d)\\ \hat{f}(\cdot)\end{pmatrix}$$

The Weyl-Titchmarsh solutions are scalar multiples of the Jost solutions $f_+(R,z)$ which are determined via the condition that

$$\mathcal{L}f_+(\cdot,z)=zf_+(\cdot,z), \quad f_+(R,z)\sim e^{i\sqrt{z}R} \ \text{ as } \ R\to\infty \tag{4.2}$$

where $\operatorname{Im} z\geq 0$, $\operatorname{Im}\sqrt{z}\geq 0$. They are solutions to the integral equation, with $V(R)=-5(1+R^2/3)^{-2}$,

$$f_+(R,z)=e^{i\sqrt{z}R}+\int_R^\infty \frac{\sin(\sqrt{z}(R'-R))}{\sqrt{z}}V(R')f_+(R',z)\,dR'$$

The functions $\psi_+(R,z)$ have well-defined limits as $z\to\xi+i0$ when $\xi>0$. In particular, $\psi_+(R,\xi)\sim c_0(\xi)e^{i\xi^{\frac{1}{2}}R}$ as $R\to\infty$. The constant $c_0(\xi)$ is determined from the Wronskian condition $W(\psi_+(\cdot,\xi),\phi(\cdot,\xi))=1$. Once we have determined $c_0(\xi)$ we find $m(\xi+i0)$ from the Wronskian relation

$$m(\xi+i0)=W(\theta(\cdot,\xi),\psi_+(\cdot,\xi+i0))$$

We now give an asymptotic expansion of our fundamental system for small z.

Proposition 4.4. *For any $z \in \mathbb{C}$ the fundamental system $\phi(R,z)$, $\theta(R,z)$ admits absolutely convergent asymptotic expansions*

$$\phi(R,z) = \phi_0(R) + R^{-1}\sum_{j=1}^{\infty}(R^2 z)^j \phi_j(R^2)$$

$$\theta(R,z) = \theta_0(R) + \sum_{j=1}^{\infty}(R^2 z)^j \theta_j(R^2)$$

where the functions ϕ_j, θ_j are holomorphic in $\Omega = \{u \in \mathbb{C} : \ \mathrm{Re}\, u > -\frac{1}{2}\}$ and satisfy the bounds

$$|\phi_j(u)| \le \frac{C^j}{(j-1)!}|u|\langle u\rangle^{-\frac{1}{2}}, \tag{4.3}$$

$$|\theta_j(u)| \le \frac{C^j}{(j-1)!}\langle u\rangle^{\frac{1}{2}}, \qquad u \in \Omega \tag{4.4}$$

Furthermore,

$$\phi_1(u) = \begin{cases} -\frac{1}{6}u(1+o(1)) & \text{as } \ u \to 0 \\ \frac{\sqrt{3}}{2}u^{\frac{1}{2}}(1+o(1)) & \text{as } \ u \to \infty \end{cases} \tag{4.5}$$

Proof. We begin with ϕ. We formally write

$$\phi(R,z) = R^{-1}\sum_{j=0}^{\infty} z^j f_j(R), \qquad f_0(R) = R\phi_0(R)$$

This becomes rigorous once we verify the convergence of the series in any reasonable sense. The functions f_j should solve

$$\mathcal{L}(R^{-1}f_j) = R^{-1}f_{j-1}, \quad f_j(0) = f_j'(0) = 0$$

where we have set $f_{-1} = 0$. The forward fundamental solution for $\mathcal{L}$ is

$$H(R,R') = (\phi_0(R)\theta_0(R') - \phi_0(R')\theta_0(R))1_{[R>R']}$$

Hence we have the iterative relation

$$f_j(R) = \int_0^R \frac{R}{R'}\Big[\phi_0(R)\theta_0(R') - \phi_0(R')\theta_0(R)\Big] f_{j-1}(R')\, dR', \quad f_0(R) = R\phi_0(R)$$

Using the expressions for ϕ_0, θ_0 we rewrite this as

$$f_j(R) = \int_0^R \frac{R^2(1-R^2/3)(1-2R'^2+R'^4/9) - RR'(1-R'^2/3)(1-2R^2+R^4/9)}{R'(1+R^2/3)^{\frac{3}{2}}(1+R'^2/3)^{\frac{3}{2}}} f_{j-1}(R')\, dR'$$

It is clear from this integral that each $f_j(R)$ is an analytic function of R^2 provided $\mathrm{Re}\, R^2 > -1$. Moreover, $f_j(R^2)$ vanishes like $R^{2(j+1)}$ around $R = 0$, and grows like R^{2j+1} as $R \to \infty$. The bounds in (4.3) are a quantitative version thereof, and proved by induction.

For $\theta(R,z)$ we make the ansatz

$$\theta(R,z) = \sum_{j=0}^{\infty} z^j g_j(R), \quad g_0(R) = \theta_0(R)$$

where the functions g_j solve

$$\mathcal{L}g_j = g_{j-1}, \qquad g_{-1} = 0$$

Thus, the iterative relation is

$$g_j(R) = \int_0^R \Big[\phi_0(R)\theta_0(R') - \phi_0(R')\theta_0(R)\Big] g_{j-1}(R')\, dR', \quad g_0(R) = \theta_0(R)$$

The analyticity is the same as for the $f_j(R)$, and $g_j(R)$ vanishes like R^{2j} around $R = 0$, whereas the growth is R^{2j+1} as $R \to \infty$, as claimed.

Finally, the leading order of $\phi_1(u)$ is found by solving for the coefficients c_1, c_2 in

$$\begin{aligned}
&(-\partial_{RR} - 5(1 + R^2/3)^{-2})c_1R^3(1 + o(1)) = R(1 + o(1)) \quad \text{as} \quad R \to 0\\
&(-\partial_{RR} - 5(1 + R^2/3)^{-2})c_2R^2(1 + o(1)) = -\sqrt{3}(1 + o(1)) \quad \text{as} \quad R \to \infty,
\end{aligned}$$

respectively. Thus, $c_1 = -\frac{1}{6}$, $c_2 = \frac{1}{2}\sqrt{3}$ as claimed. □

Next, we turn to the asymptotic expansion of the Jost solutions.

Proposition 4.5. *For any $\xi > 0$, the Jost solution $f_+(\cdot, \xi)$ as in (4.2) is of the form*

$$f_+(R, \xi) = e^{iR\xi^{\frac{1}{2}}}\sigma(R\xi^{\frac{1}{2}}, R), \qquad R^2\xi \gtrsim 1$$

where σ admits the asymptotic series approximation

$$\sigma(q, R) \approx \sum_{j=0}^{\infty} q^{-j}\psi_j^+(R)$$

in the sense that for all integers $j_0 \geq 0$, and all indices α, β, we have

$$\sup_{R>0} \langle R\rangle^2 \Big|(R\partial_R)^\alpha(q\partial_q)^\beta \Big[\sigma(q, R) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(R)\Big]\Big| \leq c_{\alpha,\beta,j_0}\, q^{-j_0-1} \tag{4.6}$$

for all $q > 1$. Here

$$\psi_0^+ = 1, \quad \psi_1^+(R) = \begin{cases} ic_1R^{-2} + iO(R^{-4}) & \text{as} \quad R \to \infty \\ ic_2R + iO(R^2) & \text{as} \quad R \to 0 \end{cases}$$

with some real constants c_1, c_2. More generally, $\psi_j^+(R)$ are smooth symbols of order -2 for $j \geq 1$, i.e., for all $k \geq 0$

$$\sup_{R>0} \langle R\rangle^2\, |(\langle R\rangle\partial_R)^k\psi_j^+(R)| < \infty$$

Finally, $\psi_j^+(R) = O(R^j)$ as $R \to 0$.

Proof. With the notation

$$\sigma(q, R) = f_+(R, \xi)e^{-iR\xi^{\frac{1}{2}}}$$

we need to solve the conjugated equation

$$\left(-\partial_{RR} - 2i\xi^{\frac{1}{2}}\partial_R - \frac{5}{(1 + R^2/3)^2}\right)\sigma(R\xi^{\frac{1}{2}}, R) = 0 \tag{4.7}$$

We look for a formal power series solving this equation,

$$\sum_{j=0}^{\infty} \xi^{-\frac{j}{2}} f_j(R) \tag{4.8}$$

This yields a recurrence relation for the f_j's,

$$2i\partial_R f_j = \left(-\partial_{RR} - \frac{5}{(1+R^2/3)^2}\right) f_{j-1}, \quad f_j(\infty) = f_j'(\infty) = 0$$

with $f_0 = 1$, which is solved by

$$f_j(R) = \frac{i}{2}\partial_R f_{j-1}(R) - \frac{i}{2}\int_R^\infty \frac{5}{(1+R'^2/3)^2} f_{j-1}(R')\, dR'$$

Then $f_1(R)$ is smooth for all $R \geq 0$ with $f_1(R) = -\frac{15i}{2R^3} + iO(R^{-5})$ as $R \to \infty$ and $f_1(R) = ic + iO(R)$ around $R = 0$. More generally, $f_j(R) = i^j O(R^{-j-2})$ as $R \to \infty$ and $f_j(R) = i^j O(1)$ as $R \to 0$. Differentiating these functions leads to symbol-type behavior as $R \to \infty$. Defining $\psi_j^+(R) := R^j f_j(R)$ yields the desired bounds.

To finish the proof, we first construct an approximate sum, i.e., a function $\sigma_{ap}(q,R)$ with the property that for each $j_0 \geq 0$ we have

$$\left|(R\partial_R)^\alpha (q\partial_q)^\beta \left[\sigma_{ap}(q,R) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(R)\right]\right| \leq c_{\alpha,\beta,j_0} \langle R\rangle^{-2} q^{-j_0-1} \tag{4.9}$$

The construction of $\sigma_{ap}(q,R)$ is standard in symbol calculus; we set

$$\sigma_{ap}(q,R) := \sum_{j=0}^\infty q^{-j}\psi_j^+(R)\chi(q\delta_j)$$

where $\delta_j \to 0$ sufficiently fast and χ is a cut-off function which vanishes around zero and is equal to one for large arguments. The bound (4.9) implies that $\sigma_{ap}(R\xi^{\frac{1}{2}}, R)$ is a good approximate solution for (4.7) at infinity, namely the error

$$e(R\xi^{\frac{1}{2}}, R) = \left(-\partial_{RR} - 2i\xi^{\frac{1}{2}}\partial_R - \frac{5}{(1+R^2/3)^2}\right)\sigma_{ap}(R,\xi)$$

satisfies for all indices α, β, j

$$|(R\partial_R)^\alpha (q\partial_q)^\beta e(q,R)| \leq c_{\alpha,\beta,j} \langle R\rangle^{-4} q^{-j}$$

To conclude the proof it remains to solve the equation for the difference $\sigma_1 = -\sigma + \sigma_{ap}$,

$$\left(-\partial_{RR} - 2i\xi^{\frac{1}{2}}\partial_R - \frac{5}{(1+R^2/3)^2}\right)\sigma_1(R\xi^{\frac{1}{2}}, R) = e(R\xi^{\frac{1}{2}}, R)$$

with zero Cauchy data at infinity. We claim that the solution σ_1 satisfies

$$|(R\partial_R)^\alpha (q\partial_q)^\beta \sigma_1(q,R)| \leq c_{\alpha,\beta,j}\, q^{-j}\langle R\rangle^{-2}, \qquad j \geq 2$$

Note that this finishes the proof by defining $\sigma = \sigma_{ap} - \sigma_1$. A change of variable allows us to switch from the pair of operators $(R\partial_R, q\partial_q)$ to $(R\partial_R, \xi\partial_\xi)$ with comparable bounds. We rewrite the above equation as a first order system for $(v_1, v_2) = (\sigma_1, R\partial_R\sigma_1)$:

$$\partial_R \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & R^{-1} \\ -\frac{5R}{(1+R^2/3)^2} & R^{-1} - 2i\xi^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Re \end{pmatrix}$$

Then we have

$$\frac{d}{dR}|v|^2 \gtrsim -R^{-1}|v|^2 - R|v||e|$$

which gives

$$\frac{d}{dR}|v| \geq -C(R^{-1}|v| + R|e|)$$

and by Gronwall

$$|v(R)| \leq \int_R^\infty \left(\frac{R'}{R}\right)^C R'|e(R')|\,dR'$$

Then for large j we have

$$|e| \lesssim \xi^{-\frac{j}{2}} R^{-j} \langle R\rangle^{-4} \implies |v| \lesssim \xi^{-\frac{j}{2}} R^{-j} \langle R\rangle^{-2} \lesssim q^{-j}\langle R\rangle^{-2} \tag{4.10}$$

To estimate derivatives of v we commute them with the operator. For derivatives with respect to R we have

$$\partial_R(R\partial_R)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{R} \\ -\frac{5R}{(1+R^2/3)^2} & \frac{1}{R} - 2i\xi^{\frac{1}{2}} \end{pmatrix} (R\partial_R)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{1}{R} \\ \frac{10R(1-R^2/3)}{(1+R^2/3)^3} & \frac{1}{R} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ R\partial_R(Re) \end{pmatrix}$$

But the right-hand side is bounded by R^{-j-1} from the previous step and the hypothesis on e, therefore as above $R\partial_R v$ is bounded by R^{-j}.
We argue similarly for the ξ derivatives. We have

$$\partial_R(\xi\partial_\xi)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{R} \\ -\frac{5R}{(1+R^2/3)^2} & \frac{1}{R} - 2i\xi^{\frac{1}{2}} \end{pmatrix} (\xi\partial_\xi)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & i\xi^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \xi\partial_\xi(Re) \end{pmatrix}$$

The only difference is in the first term on the right, for which we write $\xi^{\frac{1}{2}} = R^{-1}q$ and we use the decay property of v with j replaced by $j+1$:

$$|\xi^{\frac{1}{2}} v_2| \lesssim \xi^{\frac{1}{2}} q^{-j-1} \lesssim R^{-1}q^{-j}, \qquad |\xi\partial_\xi(Re)| \lesssim R^{-1}q^{-j}$$

as desired. Finally, higher order derivatives are estimated by induction using the above arguments at each step. $\square$

Next, we describe the spectral measure of $\mathcal{L}$. Due to the resonance at zero energy, the spectral density becomes unbounded for small ξ. In what follows, $f_-(R,\xi) := f_+(R,-\xi) = \overline{f_+(R,\xi)}$.

Lemma 4.6. *For all $\xi > 0$ there is $a(\xi) \neq 0$ so that*

$$\phi(R,\xi) = a(\xi) f_+(R,\xi) + \overline{a(\xi)} f_-(R,\xi)$$

with

$$|a(\xi)| \asymp \begin{cases} 1 & as \;\; \xi \to 0 \\ \xi^{-\frac{1}{2}} & as \;\; \xi \to \infty \end{cases}$$

with symbol type behavior of all its derivatives. The density $\rho(\xi)$ of the spectral measure satisfies

$$\rho(\xi) \asymp \begin{cases} \xi^{-\frac{1}{2}} & as \;\; \xi \to 0 \\ \xi^{\frac{1}{2}} & as \;\; \xi \to \infty \end{cases}$$

with symbol type behavior of all derivatives.

Proof. By inspection,

$$a(\xi) = \frac{W(\phi(\cdot,\xi), f_-(\cdot,\xi))}{W(f_+(\cdot,\xi), f_-(\cdot,\xi))} = \frac{1}{-2i\xi^{\frac{1}{2}}} W(\phi(\cdot,\xi), f_-(\cdot,\xi))$$

By the preceding asymptotic analysis we can evaluate

$$W(\phi(\cdot,\xi), f_+(\cdot,\xi)) = \phi(\varepsilon\xi^{-\frac{1}{2}},\xi) f_+'(\varepsilon\xi^{-\frac{1}{2}},\xi) - \phi'(\varepsilon\xi^{-\frac{1}{2}},\xi) f_+(\varepsilon\xi^{-\frac{1}{2}},\xi)$$

with some small fixed $\varepsilon > 0$ to conclude that

$$|W(\phi(\cdot,\xi), f_+(\cdot,\xi))| \lesssim \begin{cases} \xi^{\frac{1}{2}} & \text{as } \xi \to 0 \\ 1 & \text{as } \xi \to \infty \end{cases}$$

with the corresponding upper bound on the derivatives. This yields the desired upper bound on $|a(\xi)|$. To obtain the lower bound, we proceed as follows. First, observe that

$$\operatorname{Im}(f_+(R,\xi) f_-'(R,\xi)) = -\xi^{\frac{1}{2}}$$

Second, it follows that

$$\operatorname{Im}(f_-'(R,\xi) W(f_+(R,\xi), \phi(R,\xi))) = \xi^{\frac{1}{2}} \phi'(R,\xi)$$

so that

$$|W(f_+(R,\xi), \phi(R,\xi))| \geq \xi^{\frac{1}{2}} \frac{|\phi'(R,\xi)|}{|f_+'(R,\xi)|}$$

¿From our asymptotic analysis, again at $R = \varepsilon\xi^{-\frac{1}{2}}$,

$$\frac{|\phi'(R,\xi)|}{|f_+'(R,\xi)|} \gtrsim \begin{cases} 1 & \text{as } \xi \to 0 \\ \xi^{-\frac{1}{2}} & \text{as } \xi \to \infty \end{cases}$$

which leads to the claimed lower bound on $|W(\phi(\cdot,\xi), f_+(\cdot,\xi))|$.
The Weyl solution

$$\psi_+(R, \xi + i0) = \theta(R,\xi) + m(\xi + i0)\phi(R,\xi)$$

satisfies $\psi_+(\cdot, \xi + i0) = c_0(\xi) f_+(\cdot,\xi)$. Solving for $c_0(\xi)$ yields

$$m(\xi + i0) = \frac{W(\theta(\cdot,\xi), f_+(\cdot,\xi))}{W(f_+(\cdot,\xi), \phi(\cdot,\xi))} = \frac{W(\theta(\cdot,\xi), f_+(\cdot,\xi)) W(f_-(\cdot,\xi), \phi(\cdot,\xi))}{|W(f_+(\cdot,\xi), \phi(\cdot,\xi))|^2}$$

Since

$$f_+(\cdot,\xi) = -\phi(\cdot,\xi) W(f_+(\cdot,\xi), \theta(\cdot,\xi)) + \theta(\cdot,\xi) W(f_+(\cdot,\xi), \phi(\cdot,\xi))$$

implies that

$$-2i\xi^{\frac{1}{2}} = W(f_+(\cdot,\xi), f_-(\cdot,\xi)) = -2i \operatorname{Im}\left[W(\theta(\cdot,\xi), f_+(\cdot,\xi)) W(f_-(\cdot,\xi), \phi(\cdot,\xi))\right]$$

we conclude that

$$\rho(\xi) = \frac{1}{\pi} \operatorname{Im} m(\xi + i0) = \frac{\xi^{\frac{1}{2}}}{\pi |W(f_+(\cdot,\xi), \phi(\cdot,\xi))|^2} = \frac{1}{4\pi} \frac{1}{\xi^{\frac{1}{2}} |a(\xi)|^2} \tag{4.11}$$

The denominator was estimated above, leading to the desired bound on the spectral density. □

5. The transference identity

Returning to the radiation part $\tilde{\epsilon}$ in (3.5), the idea is to expand it in terms of the generalized Fourier basis $\phi(R,\xi)$ from Proposition 4.3, i.e., write

$$\tilde{\varepsilon}(\tau,R) = x_0(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

and deduce a transport equation for the Fourier coefficients $(x_0(\tau), x(\tau,\xi))$. The main difficulty in doing this is caused by the operator $R\partial_R$ which is not diagonal in the Fourier basis. We re-express this derivative in terms of the derivative $2\xi\partial_\xi$. We refer to this procedure, which involves a certain error operator, as the transference identity since it allows us to transfer derivatives from R to ξ. We define the error operator $\mathcal{K}$ by

$$\widehat{R\partial_R u} = -2\xi\partial_\xi\hat{u} + \mathcal{K}\hat{u} \tag{5.1}$$

where $\hat{f} = \mathcal{F}f$ is the "distorted Fourier transform" from Proposition 4.3 and the operator $-2\xi\partial_\xi$ acts only on the continuous part of the spectrum. Apriori we have

$$\mathcal{K} : C_0^\infty(\{\xi_d\}\cup(0,\infty)) \to C^\infty(\{\xi_d\}\cup(0,\infty))$$

Splitting functions on $\mathrm{spec}(\mathcal{L})$ into a discrete and continuous component we obtain a matrix representation for $\mathcal{K}$,

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{dd} & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_{cc} \end{pmatrix}$$

Using the expressions for the direct and inverse Fourier transform in Proposition 4.3 we obtain

$$\begin{aligned}
\mathcal{K}_{dd} &= \Big\langle R\partial_R\phi_d(R), \phi_d(R)\Big\rangle_{L^2_R} \\
\mathcal{K}_{dc}f &= \Big\langle \int_0^\infty f(\xi)R\partial_R\phi(R,\xi)\rho(\xi)\,d\xi\,,\ \phi_d(R)\Big\rangle_{L^2_R} \\
\mathcal{K}_{cd}(\eta) &= \Big\langle R\partial_R\phi_d, \phi(R,\eta)\Big\rangle_{L^2_R} \\
\mathcal{K}_{cc}f(\eta) &= \Big\langle \int_0^\infty f(\xi)R\partial_R\phi(R,\xi)\rho(\xi)\,d\xi\,,\ \phi(R,\eta)\Big\rangle_{L^2_R} \\
&\quad + \Big\langle \int_0^\infty 2\xi\partial_\xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi\,,\ \phi(R,\eta)\Big\rangle_{L^2_R}
\end{aligned} \tag{5.2}$$

Integrating by parts with respect to R in the first two relations we obtain

$$\mathcal{K}_{dd} = -\frac{1}{2}, \qquad \mathcal{K}_{dc}f = -\int_0^\infty f(\xi)K_d(\xi)\rho(\xi)d\xi, \qquad \mathcal{K}_{cd}(\eta) = K_d(\eta)$$

where

$$K_d(\eta) = \Big\langle R\partial_R\phi_d, \phi(R,\eta)\Big\rangle_{L^2_R}$$

Integrating by parts with respect to ξ in (5.2) yields

$$\begin{aligned}
\mathcal{K}_{cc}f(\eta) &= \Big\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi\,,\ \phi(R,\eta)\Big\rangle_{L^2_R} \\
&\quad - 2\left(1 + \frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta)
\end{aligned} \tag{5.3}$$

where the scalar product is to be interpreted in the principal value sense with $f \in C_0^\infty((0,\infty))$.

In this section, we study the boundedness properties of the operator $\mathcal{K}$. We begin with a description of the function K_d and of the kernel $K_0(\eta,\xi)$ of $\mathcal{K}_{cc}$.

Theorem 5.1. *a) The operator $\mathcal{K}_{cc}$ can be written as*

$$\mathcal{K}_{cc} = -\Big(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho(\eta)}\Big)\delta(\xi-\eta) + \mathcal{K}_0 \tag{5.4}$$

where the operator $\mathcal{K}_0$ has a kernel $K_0(\eta,\xi)$ of the form[2]

$$K_0(\eta,\xi) = \frac{\rho(\xi)}{\eta-\xi}F(\xi,\eta) \tag{5.5}$$

with a symmetric function $F(\xi,\eta)$ of class C^2 in $(0,\infty)\times(0,\infty)$ satisfying the bounds

$$|F(\xi,\eta)| \lesssim \begin{cases} \xi+\eta & \xi+\eta\leq 1 \\ (\xi+\eta)^{-1}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\geq 1 \end{cases}$$

$$|\partial_\xi F(\xi,\eta)| + |\partial_\eta F(\xi,\eta)| \lesssim \begin{cases} 1 & \xi+\eta\leq 1 \\ (\xi+\eta)^{-\frac32}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\geq 1 \end{cases}$$

$$\sup_{j+k=2} |\partial_\xi^j\partial_\eta^k F(\xi,\eta)| \lesssim \begin{cases} (\xi+\eta)^{-\frac12} & \xi+\eta\leq 1 \\ (\xi+\eta)^{-2}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\geq 1 \end{cases}$$

where N is an arbitrary large integer.

b) The function K_d is smooth and rapidly decaying at infinity.

Proof. We first establish the off-diagonal behavior of $\mathcal{K}_{cc}$, and later return to the issue of identifying the δ-measure that sits on the diagonal. We begin with (5.3) with $f\in C_0^\infty((0,\infty))$. The integral

$$u(R) = \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi$$

behaves like R at 0 and is a Schwartz function at infinity. The second factor $\phi(R,\eta)$ in (5.3) also decays like R at 0 but at infinity it is only bounded with bounded derivatives. Then the following integration by parts is justified:

$$\eta\mathcal{K}_{cc}f(\eta) = \Big\langle u, \mathcal{L}\phi(R,\eta)\Big\rangle_{L^2_R} = \Big\langle \mathcal{L}u, \phi(R,\eta)\Big\rangle_{L^2_R}$$

Moreover,

$$\begin{aligned}\mathcal{L}u &= \int_0^\infty f(\xi)[\mathcal{L},R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty f(\xi)(R\partial_R - 2\xi\partial_\xi)\xi\phi(R,\xi)\rho(\xi)\,d\xi \\ &= \int_0^\infty f(\xi)[\mathcal{L},R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi \\ &\quad - 2\int_0^\infty \xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi\end{aligned}$$

with the commutator

$$[\mathcal{L},R\partial_R] = 2\mathcal{L} + \frac{10}{(1+R^2/3)^2} - \frac{20R^2}{3(1+R^2/3)^3} =: 2\mathcal{L} + U(R)$$

[2]The kernel below is interpreted in the principal value sense

Thus,

$$\mathcal{L}u = \int_0^\infty f(\xi)U(R)\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

Hence we obtain

$$\eta\mathcal{K}_{cc}f(\eta) - \mathcal{K}_{cc}(\xi f)(\eta) = \Big\langle \int_0^\infty f(\xi)U(R)\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\Big\rangle_{L^2_R}$$

The double integral on the right-hand side is absolutely convergent, therefore we can change the order of integration to obtain

$$(\eta-\xi)K_0(\eta,\xi) = \rho(\xi)\Big\langle U(R)\phi(R,\xi), \phi(R,\eta)\Big\rangle_{L^2_R}$$

This leads to the representation in (5.5) when $\xi \neq \eta$ with

$$F(\xi,\eta) = \Big\langle U(R)\phi(R,\xi), \phi(R,\eta)\Big\rangle_{L^2_R}$$

It remains to study its size and regularity. First, due to our pointwise bound from the previous section,

$$\begin{aligned}
\sup_{R\geq 0}|\phi(R,\xi)| &\lesssim \langle\xi\rangle^{-\frac12}, \\
|R\partial_R\phi(R,\xi)| &\lesssim R \qquad \forall\ \xi>1 \\
|\partial_\xi\phi(R,\xi)| &\lesssim \min(R\xi^{-1}, R^3) \qquad \forall\ \xi>1 \\
|\partial_\xi\phi(R,\xi)| &\lesssim \min(R\xi^{-\frac12}, R^2) \qquad \forall\ 0<\xi<1 \\
|\partial_\xi^2\phi(R,\xi)| &\lesssim \min(R^2\xi^{-\frac32}, R^5) \qquad \forall\ \xi>1 \\
|\partial_\xi^2\phi(R,\xi)| &\lesssim \min(R^2\xi^{-1}, R^4) \qquad \forall\ 0<\xi<1
\end{aligned} \tag{5.6}$$

we always have the estimates

$$\begin{aligned}
|F(\xi,\eta)| &\lesssim \langle\xi\rangle^{-\frac12}\langle\eta\rangle^{-\frac12}, \\
|\partial_\xi F(\xi,\eta)| &\lesssim \langle\xi\rangle^{-1}\langle\eta\rangle^{-\frac12}, \quad |\partial_\eta F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac12}\langle\eta\rangle^{-1}, \\
|\partial_{\xi\eta}F(\xi,\eta)| &\lesssim \xi^{-1}\eta^{-1} \qquad \forall\ \xi>1,\ \eta>1 \\
|\partial_\xi^2 F(\xi,\eta)| &\lesssim \xi^{-\frac32}\eta^{-\frac12} \qquad \forall\ \xi>1,\ \eta>1 \\
|\partial_\eta^2 F(\xi,\eta)| &\lesssim \xi^{-\frac12}\eta^{-\frac32} \qquad \forall\ \xi>1,\ \eta>1
\end{aligned} \tag{5.7}$$

They are only useful when ξ and η are very close. To improve on them, we consider two cases:

Case 1: $1 \lesssim \xi+\eta$**.** To capture the cancelations when ξ and η are separated we resort to another integration by parts,

$$\eta F(\xi,\eta) = \Big\langle U(R)\phi(R,\xi), \mathcal{L}\phi(R,\eta)\Big\rangle = \Big\langle [\mathcal{L}, U(R)]\phi(R,\xi), \phi(R,\eta)\Big\rangle + \xi F(\xi,\eta) \tag{5.8}$$

Hence, evaluating the commutator,

$$(\eta-\xi)F(\xi,\eta) = -\Big\langle (2U_R\partial_R + U_{RR})\phi(R,\xi), \phi(R,\eta)\Big\rangle \tag{5.9}$$

Since $U_R(0)=0$ it follows that $(2U_R\partial_R + U_{RR})\phi(R,\xi)$ has the same behavior as $\phi(R,\xi)$ in the first region. Then we can repeat the argument above to obtain

$$(\eta-\xi)^2F(\xi,\eta) = -\Big\langle [\mathcal{L}, 2U_R\partial_R + U_{RR}]\phi(R,\xi), \phi(R,\eta)\Big\rangle$$

The second commutator has the form, with $V(R) := -5(1+R^2/3)^{-2}$,

$$[\mathcal{L}, 2U_R\partial_R + U_{RR}] = 4U_{RR}\mathcal{L} - 4U_{RRR}\partial_R - U_{RRRR} - 2U_RV_R - 4U_{RR}V$$

Since $V(R), U(R)$ are even, this leads to

$$(\eta-\xi)^2 F(\xi,\eta) = \Big\langle (U^{odd}(R)\partial_R + U^{even}(R) + \xi U^{even}(R))\phi(R,\xi), \phi(R,\eta)\Big\rangle$$

where by U^{odd}, respectively U^{even}, we have generically denoted odd, respectively even, nonsingular rational functions with good decay at infinity. Inductively, one now verifies the identity

$$\begin{aligned}(\eta-\xi)^{2k} F(\xi,\eta) &= \Big\langle \Big(\sum_{j=0}^{k-1} \xi^j\, U^{odd}_{kj}\, \partial_R + \sum_{\ell=0}^{k} \xi^\ell U^{even}_{k\ell}\Big)\phi(\cdot,\xi), \phi(\cdot,\eta)\Big\rangle \\ \langle R\rangle |U^{odd}_{kj}(R)| &+ |U^{even}_{k\ell}(R)| \lesssim \langle R\rangle^{-4-2k} \qquad \forall\, j,\ell\end{aligned} \tag{5.10}$$

By means of the pointwise bounds on ϕ and $\partial_R\phi$ from (5.6) we infer from this that

$$|F(\xi,\eta)| \lesssim \frac{\langle\xi\rangle^{k-\frac12}\langle\eta\rangle^{-\frac12}}{(\eta-\xi)^{2k}} \qquad \forall\,\xi,\ \eta>0$$

Combining this estimate with (5.7) yields, for arbitrary N,

$$|F(\xi,\eta)| \lesssim (\xi+\eta)^{-1}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} \ \text{ provided }\ \xi+\eta \gtrsim 1,$$

as claimed. For the derivatives of F we follow a similar procedure. If ξ and η are comparable, then from (5.7), $|\partial_\eta F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac32}$. Otherwise we differentiate with respect to η in (5.10). This yields

$$\begin{aligned}(\eta-\xi)^{2k}\partial_\eta F(\xi,\eta) = \Big\langle \Big(\sum_{j=0}^{k-1} \xi^j\, U^{odd}_{kj}\, \partial_R + \sum_{\ell=0}^{k} \xi^\ell U^{even}_{k\ell}\Big)\phi(R,\xi), \partial_\eta\phi(R,\eta)\Big\rangle \\ - 2k(\eta-\xi)^{2k-1}F(\xi,\eta)\end{aligned}$$

Using also the bound on F from above we obtain

$$|\partial_\eta F(\xi,\eta)| \lesssim \frac{\xi^{k-\frac12}\eta^{-1} + \xi^{k-1}\eta^{-\frac12}}{(\eta-\xi)^{2k}}, \qquad 1\lesssim \xi,\eta$$

respectively

$$|\partial_\eta F(\xi,\eta)| \lesssim \frac{\eta^{-1}}{(\eta-\xi)^{2k}} \qquad \xi \ll 1 \lesssim \eta$$

and

$$|\partial_\eta F(\xi,\eta)| \lesssim \frac{\xi^{k-\frac12}}{(\eta-\xi)^{2k}} \qquad \eta \ll 1 \lesssim \xi$$

which again yield the desired bounds. Finally, we consider the second order derivatives with respect to ξ and η. For ξ and η close we again use the bound from (5.7). Otherwise we differentiate twice in (5.10) and continue as before. We note that it is important here that the decay of U^{odd}_{kj} and $U^{even}_{k\ell}$ improves with k. This is because the second order derivative bound at 0 has a sizeable growth at infinity which has to be canceled,

$$|\partial^2_\xi \phi(R,0)| \approx R^4$$

Case 2: $\xi, \eta \ll 1$. First, we note that $F(0,0) = 0$. This can be verified by direct integration, and is heuristically justified by the fact that $U = [\mathcal{L}, R\partial_R] - 2\mathcal{L}$. The pointwise bound

$$\sup_{\xi,\eta>0} |\partial_\xi F(\xi,\eta)| \lesssim 1$$

follows by differentiating (5.8) and from the bound $|\partial_\xi \phi(R,\xi)| \lesssim R^2$, see (5.6). To bound the second order derivatives of F we recall the pointwise bounds, for $0 < \xi < 1$,

$$|\partial_\xi^j \phi(R,\xi)| \lesssim \left\{ \begin{array}{cc} R^{2j} & R < \xi^{-\frac12} \\ \xi^{-j/2}\, R^j & R \geq \xi^{-\frac12} \end{array} \right. , \qquad j = 0,1,2$$

If $\eta < \xi < 1$, then these bounds imply that

$$\begin{aligned} |\partial_{\xi\eta} F(\xi,\eta)| &\lesssim \int_0^{\xi^{-\frac12}} \langle R\rangle^{-4} R^4 \, dR + \int_{\xi^{-\frac12}}^{\eta^{-\frac12}} \langle R\rangle^{-4} R^3 \xi^{-\frac12} \, dR + \int_{\eta^{-\frac12}}^\infty \langle R\rangle^{-2} (\xi\eta)^{-\frac12} \, dR \\ &\lesssim [1 + \log(\xi/\eta)]\xi^{-\frac12} \end{aligned}$$

The logarithm in the middle integral is an artefact and can be removed using the oscillatory nature of $\partial_\xi \phi(R,\xi)$ in the regime $R^2\xi > 1$ as provided by Proposition 4.5 and Lemma 4.6. Loosely speaking, this means integrating by parts using that $\partial_\xi \phi(R,\xi) \sim R\xi^{-1} \partial_R e^{iR\xi^{\frac12}}$ for $R^2\xi > 1$ and small ξ. Thus, actually

$$|\partial_{\xi\eta} F(\xi,\eta)| \lesssim \xi^{-\frac12}$$

A similar computation yields, for $\xi < 1$,

$$|\partial_\xi^2 F(\xi,\eta)| \lesssim \int_0^{\xi^{-\frac12}} \langle R\rangle^{-4} R^4 \, dR + \int_{\xi^{-\frac12}}^\infty \langle R\rangle^{-4} R^2 \xi^{-1} \, dR \lesssim \xi^{-\frac12}$$

This bound is too weak when $\xi \ll \eta < 1$. In that case, we differentiate (5.9) to obtain

$$(\eta - \xi)\partial_\xi^2 F(\xi,\eta) = 2\partial_\xi F(\xi,\eta) + \Big\langle \partial_\xi^2 \phi(R,\xi), (2U_R \partial_R + U_{RR})\phi(R,\eta) \Big\rangle$$

which in turn yields
(5.11)

$$(\eta - \xi)\partial_\xi^2 F(\xi,\eta) = \int_\xi^\eta \Big[2\partial_{\xi\zeta} F(\xi,\zeta) + \Big\langle \partial_\xi^2 \phi(\cdot,\xi), (2U_R \partial_R + U_{RR})\partial_\zeta \phi(\cdot,\zeta) \Big\rangle \Big] \, d\zeta$$

Using also the bound

$$|R\partial_{R\zeta} \phi(R,\zeta)| \lesssim \min(R\zeta^{-\frac12}, R^2)$$

we can evaluate the inner product in (5.11) as follows:

$$\begin{aligned} &\Big| \Big\langle \partial_\xi^2 \phi(\cdot,\xi), (2U_R \partial_R + U_{RR})\partial_\zeta \phi(\cdot,\zeta) \Big\rangle \Big| \\ &\lesssim \int_0^{\zeta^{-\frac12}} \langle R\rangle^{-6} R^4 R^2 \, dR + \int_{\zeta^{-\frac12}}^{\xi^{-\frac12}} \langle R\rangle^{-6} R^4 R \zeta^{-\frac12} \, dR + \int_{\xi^{-\frac12}}^\infty \langle R\rangle^{-6} R^2 \xi^{-1} R \zeta^{-\frac12} \, dR \\ &\lesssim [1 + \log(\zeta/\xi)]\zeta^{-\frac12} \end{aligned}$$

The logarithm appearing in the middle integral can be removed as before exploiting cancellations. Thus, (5.11) is controlled by

$$|(\eta-\xi)\partial_\xi^2 F(\xi,\eta)| \lesssim \Big|\int_\xi^\eta \zeta^{-\frac12}\,d\zeta\Big| \lesssim \eta^{\frac12}$$

Since $\eta \gg \xi$ this yields

$$|\partial_\xi^2 F(\xi,\eta)| \lesssim \eta^{-\frac12}$$

which concludes the analysis of the off-diagonal part of the kernel.

Next, we extract the δ measure that sits on the diagonal of the kernel of $\mathcal{K}_{cc}$ from the representation formula (5.3), see also (5.4). To do so, we can restrict ξ,η to a compact subset of $(0,\infty)$. This is convenient, as we then have the following asymptotics of $\phi(R,\xi)$ for $R\xi^{\frac12} \gg 1$:

$$\phi(R,\xi) = 2\,\mathrm{Re}\,[a(\xi)e^{iR\xi^{\frac12}}] + O(R^{-2})$$

$$(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi) = -4\,\mathrm{Re}\,[\xi a'(\xi)e^{iR\xi^{\frac12}}] + O(R^{-2})$$

where the $O(\cdot)$ terms depend on the choice of the compact subset. The R^{-2} terms are integrable so they contribute a bounded kernel to the inner product in (5.3). The same applies to the contribution of a bounded R region. Using the above expansions, we conclude that the δ-measure contribution of the inner product in (5.3) can only come from one of the following integrals:

$$-4\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\mathrm{Re}\,\big[\xi a'(\xi)a(\eta)e^{iR(\xi^{\frac12}+\eta^{\frac12})}\big]\rho(\xi)\,d\xi dR \tag{5.12}$$

$$-4\int_0^\infty\int_0^\infty f(\xi)\chi(R)\;\mathrm{Re}\,\big[\xi a'(\xi)\bar a(\eta)e^{iR(\xi^{\frac12}-\eta^{\frac12})}\big]\;\rho(\xi)\,d\xi dR \tag{5.13}$$

where χ is a smooth cutoff function which equals 0 near $R=0$ and 1 near $R=\infty$. In all of the above integrals we can argue as in the proof of the classical Fourier inversion formula to change the order of integration. Integrating by parts in the first integral (5.12) reveals that it cannot contribute a δ-measure. On the other hand, (5.13) contributes both a Hilbert transform type kernel as well as a δ-measure to K. By inspection, the δ contribution is

$$\begin{aligned}
&-2\int_{-\infty}^\infty \mathrm{Re}\,[\xi a'(\xi)\bar a(\eta)e^{iR(\xi^{\frac12}-\eta^{\frac12})}]\rho(\xi)\,dR\\
&= -4\pi\,\mathrm{Re}\,[\xi a'(\xi)\bar a(\eta)]\rho(\xi)\delta(\xi^{\frac12}-\eta^{\frac12})\\
&= -8\pi\xi^{\frac12}\,\mathrm{Re}\,[\xi a'(\xi)\bar a(\xi)]\rho(\xi)\delta(\xi-\eta)\\
&= \Big[\frac12 + \frac{\xi\rho'(\xi)}{\rho(\xi)}\Big]\delta(\xi-\eta)
\end{aligned}$$

where we used that $\rho(\xi)^{-1} = 4\pi\xi^{\frac12}|a(\xi)|^2$ in the final step, see (4.11). Combining this with the δ-measure in (5.3) yields (5.4).

b) Arguing as in part (a) we have

$$K_d(\eta) = \frac{F(\xi_d,\eta)}{\xi_d-\eta}$$

For F we use the representation in (5.10) with ξ replaced by ξ_d and $\phi(\cdot,\xi)$ replaced by ϕ_d. The conclusion easily follows from pointwise bounds on $\phi(\cdot,\eta)$ and its derivatives. $\square$

Next we consider the L^2 mapping properties for $\mathcal{K}$. We introduce the weighted L^2 spaces $L^{2,\alpha}_\rho$ of functions on spec$(\mathcal{L})$ with norm

$$\|f\|^2_{L^{2,\alpha}_\rho} := |f(\xi_d)|^2 + \int_0^\infty |f(\xi)|^2 \langle\xi\rangle^{2\alpha} \rho(\xi)\, d\xi \tag{5.14}$$

Then we have

Proposition 5.2. *a) The operators $\mathcal{K}_0$, $\mathcal{K}$ map*

$$\mathcal{K}_0 \;:\; L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho, \qquad \mathcal{K} \;:\; L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho.$$

b) In addition, we have the commutator bound

$$[\mathcal{K}, \xi\partial_\xi] \;:\; L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho$$

with $\xi\partial_\xi$ acting only on the continuous spectrum. Both statements hold for all $\alpha \in \mathbb{R}$.

Proof. We commence with the $\mathcal{K}_0$ part. a) The first property is equivalent to showing that the kernel

$$\rho^{\frac12}(\eta)\langle\eta\rangle^{\alpha+1/2} K_0(\eta,\xi)\langle\xi\rangle^{-\alpha}\rho^{-\frac12}(\xi) \;:\; L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$$

With the notation of the previous theorem, the kernel on the left-hand side is

$$\tilde K_0(\eta,\xi) := \langle\eta\rangle^{\alpha+1/2}\langle\xi\rangle^{-\alpha} \frac{\sqrt{\rho(\xi)\rho(\eta)}}{\xi-\eta} F(\xi,\eta)$$

We first separate the diagonal and off-diagonal behavior of $\tilde K_0$, considering several cases.

Case 1: $(\xi,\eta) \in Q := [0,4]\times[0,4]$**.**

We cover the unit interval with dyadic subintervals $I_j = [2^{j-1}, 2^{j+1}]$. We cover the diagonal with the union of squares

$$A = \bigcup_{j=-\infty}^{2} I_j \times I_j$$

and divide the kernel $\tilde K_0$ into

$$1_Q \tilde K_0 = 1_{A\cap Q}\tilde K_0 + 1_{Q\setminus A}\tilde K_0$$

Case 1(a): Here we show that the diagonal part $1_{A\cap Q}\tilde K_0$ of $\tilde K_0$ maps L^2 to L^2. By orthogonality it suffices to restrict ourselves to a single square $I_j \times I_j$. We recall the $T1$ theorem for Calderon-Zygmund operators, see page 293 in [19]: suppose the kernel $K(\eta,\xi)$ on $\mathbb{R}^2$ defines an operator $T : \mathcal{S} \to \mathcal{S}'$ and has the following pointwise properties with some $\gamma \in (0,1]$ and a constant C_0:

(i) $|K(\eta,\xi)| \le C_0|\xi-\eta|^{-1}$
(ii) $|K(\eta,\xi) - K(\eta',\xi)| \le C_0|\eta-\eta'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\eta-\eta'| < |\xi-\eta|/2$
(iii) $|K(\eta,\xi) - K(\eta,\xi')| \le C_0|\xi-\xi'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\xi-\xi'| < |\xi-\eta|/2$

If in addition T has the restricted L^2 boundedness property, i.e., for all $r > 0$ and $\xi_0, \eta_0 \in \mathbb{R}$, $\|T(\omega^{r,\xi_0})\|_2 \le C_0 r^{\frac12}$ and $\|T^*(\omega^{r,\eta_0})\|_2 \le C_0 r^{\frac12}$ where $\omega^{r,\xi_0}(\xi) = \omega((\xi-\xi_0)/r)$ with a fixed bump-function ω, then T and T^* are $L^2(\mathbb{R})$ bounded with an operator norm that only depends on C_0.

Within the square $I_j \times I_j$, Theorem 5.1 shows that the kernel of $\tilde K_0$ satisfies these properties with $\gamma = 1$, and is thus bounded on L^2.

Case 1(b): Consider now the off-diagonal part $1_{Q\setminus A}\tilde{K}_0$. In this region, by Theorem 5.1,

$$|\tilde{K}_0(\eta,\xi)| \lesssim (\xi\eta)^{-\frac{1}{4}}$$

which is a Hilbert-Schmidt kernel on Q and thus L^2 bounded.

Case 2: $(\xi,\eta) \in Q^c$. We cover the diagonal with the union of squares

$$B = \bigcup_{j=1}^{\infty} I_j \times I_j$$

and divide the kernel $\tilde{K}_0$ into

$$1_{Q^c}\tilde{K}_0 = 1_{B\cap Q^c}\tilde{K}_0 + 1_{Q^c\setminus B}\tilde{K}_0$$

Case 2a: Here we consider the estimate on B. As in case 1a) above, we use Calderon-Zygmund theory. Evidently, $|\tilde{K}_0(\eta,\xi)| \lesssim |\xi-\eta|^{-1}$ on B by Theorem 5.1. To check (ii) and (iii), we differentiate $\tilde{K}_0$. It will suffice to consider the case where the ∂_ξ derivative falls on $F(\xi,\eta)$. We distinguish two cases: if $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| \leq 1$, then $|\xi-\eta| \lesssim \xi^{\frac{1}{2}}$ which implies that

$$\frac{\xi^{-\frac{1}{2}}|\xi-\xi'|}{|\xi-\eta|} \lesssim \frac{|\xi-\xi'|^{\frac{1}{2}}}{|\xi-\eta|^{\frac{3}{2}}} \qquad \forall\, |\xi-\xi'| < |\xi-\eta|/2$$

if, on the other hand, $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| > 1$, then

$$\frac{\xi^{-\frac{1}{2}}|\xi-\xi'|}{|\xi-\eta||\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|} \lesssim \frac{|\xi-\xi'|}{|\xi-\eta|^2} \qquad \forall\, |\xi-\xi'| < |\xi-\eta|/2$$

which proves property (iii) on B with $\gamma = \frac{1}{2}$, and by symmetry also (ii). The restricted L^2 property follows form the cancelation in the kernel and the previous bounds on the kernel. Hence, $\tilde{K}_0$ is L^2 bounded on B.

Case 2b: Finally, in the exterior region $Q^c \setminus B$ we have the bound, with arbitrarily large N,

$$|\tilde{K}_0(\eta,\xi)| \lesssim (1+\xi)^{-N}(1+\eta)^{-N}$$

which is L^2 bounded by Schur's lemma.

This concludes the proof of the first mapping property in part (a). The second one follows in a straightforward manner since K_d is rapidly decaying at ∞.

b) A direct computation shows that the kernel K_0^{com} of the commutator $[\xi\partial_\xi, K_0]$ is given by

$$K_0^{com}(\eta,\xi) = (\eta\partial_\eta + \xi\partial_\xi)K_0(\eta,\xi) + K_0(\eta,\xi) = \frac{\rho(\xi)}{\xi-\eta}F^{com}(\xi,\eta)$$

interpreted in the principal value sense and with F^{com} given by

$$F^{com}(\xi,\eta) = \frac{\xi\rho'(\xi)}{\rho(\xi)}F(\xi,\eta) + (\xi\partial_\xi + \eta\partial_\eta)F(\xi,\eta)$$

By Theorem 5.1 this satisfies the same pointwise off-diagonal bounds as F. Near the diagonal the bounds for F^{com} and its derivatives are worse[3] than those for F by a factor of $(1+\xi)^{\frac{1}{2}}$. Then the proof of the L^2 commutator bound for K_0 is similar to the argument in part (a).

The remaining part of the commutator $[\mathcal{K}, \xi\partial_\xi]$ involves

[3]The one derivative loss can be avoided by a more careful analysis, but this does not seem necessary here.

(i) The commutator of the diagonal part of $\mathcal{K}_{cc}$ with $\xi\partial_\xi$. This is the multiplication operator by

$$\xi\partial_\xi \frac{\xi\rho'(\xi)}{\rho(\xi)}$$

which is bounded since ρ has symbol like behavior both at 0 and at ∞.

(ii) The operator $\xi\partial_\xi \mathcal{K}_{cd}$ which is given by the bounded rapidly decreasing function $\xi\partial_\xi K_d(\xi)$.

(iii) The operator $\mathcal{K}_{dc}\xi\partial_\xi$ given by

$$\mathcal{K}_{dc}\xi\partial_\xi f = \int_0^\infty K_d(\xi)\xi\partial_\xi f(\xi)d\xi = -\int_0^\infty f(\xi)\partial_\xi(\xi K_d(\xi))d\xi$$

which is also bounded due to the properties of K_d. $\square$

6. The final system of equations

We now rewrite the main equation (3.5) in terms of the Fourier transform. With $\mathcal{F}$ as in Proposition 4.3, and with $\beta = \dot\lambda\lambda^{-1}$,

$$\mathcal{F}\Big(\partial_\tau + \beta(\tau)R\partial_R\Big) = \Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)\mathcal{F} + \beta\mathcal{K}\mathcal{F}$$

which gives

$$\begin{aligned}\mathcal{F}\Big(\partial_\tau + \beta R\partial_R\Big)^2 &= \Big(\partial_\tau + \beta(-2\xi\partial_\xi + \mathcal{K})\Big)^2\mathcal{F}\\ =&\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)^2\mathcal{F} + 2\beta\mathcal{K}\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)\mathcal{F} + \beta^2(\mathcal{K}^2 + 2[\xi\partial_\xi, \mathcal{K}])\mathcal{F}\end{aligned}$$

Recall that

$$\tilde\varepsilon(\tau, R) = x_0(\tau)\phi_d + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

This leads to a transport type equation for the Fourier transform

$$X(\tau) = (x_0(\tau), x(\tau,\xi))$$

of $\tilde\varepsilon$ by applying $\mathcal{F}$ to (3.5). It is convenient to write it as a system for the two components:

$$\begin{aligned}&\begin{pmatrix}\partial_\tau^2 + \xi_d & 0\\ 0 & \Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)^2 + \xi\end{pmatrix}X\\ &= \beta(I - 2\mathcal{K})\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)X - \beta^2(\mathcal{K}^2 - \mathcal{K} + 2[\xi\partial_\xi, \mathcal{K}])X\\ &\quad + \lambda^{-2}\mathcal{F}R(N_{2k-1}(R^{-1}\mathcal{F}^{-1}X) + e_{2k-1})\end{aligned}\tag{6.1}$$

where it is understood that

$$N_{2k-1}(R^{-1}\mathcal{F}^{-1}X) + e_{2k-1} = (N_{2k-1}(R^{-1}\mathcal{F}^{-1}X) + e_{2k-1})(t(\tau), \lambda^{-1}(\tau)R)$$

Note that N_{2k-1} and e_{2k-1} are only defined on $R \lesssim \tau$, but for the Fourier transform we need to extend them to all R – this will be described in the next section, but for the moment just take an arbitrary compactly supported extension with the same regularity.

We treat problem (6.1) iteratively, as a small perturbation of the linear equation governed by the operator on the left–hand side. For this we need to solve the following uncoupled system consisting of an **elliptic equation** and a **transport equation**:

$$
\begin{aligned}
&\Big[\partial_\tau^2 + \xi_d\Big] x_d(\tau) = b_d(\tau), \\
&\Big[\Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)^2 + \xi\Big] x(\tau,\xi) = b(\tau,\xi),
\end{aligned}
\tag{6.2}
$$

We want to obtain solutions to (6.1) which decay as $\tau \to \infty$. For the first equation above this is achieved by using the standard fundamental solution H_0 which has kernel

$$H_0(\tau, s) = -\frac{1}{2}|\xi_d|^{-\frac{1}{2}} e^{-|\xi_d|^{\frac{1}{2}}|\tau-\sigma|}$$

This means that up to homogeneous solutions of the form $e^{-|\xi_d|^{\frac{1}{2}}\tau}$ the unique bounded solution to the elliptic equation is

$$x_d(\tau) = -\frac{1}{2}|\xi_d|^{-\frac{1}{2}} \int_0^\infty e^{-|\xi_d|^{\frac{1}{2}}|\tau-\sigma|}\, b_d(\sigma)\, d\sigma$$

The second equation is restricted to the range $\xi > 0$. Hence it is hyperbolic in nature, which means we can solve it backward in time, i.e., with zero Cauchy data at $\tau = \infty$. We denote by H the backward fundamental solution for the operator

$$\Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)^2 + \xi$$

and by $H(\tau,\sigma)$ its kernel,

$$x(\tau) = \int_\tau^\infty H(\tau,\sigma) f(\sigma)\, d\sigma$$

Combining the two components we obtain a fundamental solution for the system,

$$\mathcal{H} = \operatorname{diag}(H_0, H)$$

Then we look for a solution X to (6.1) as a solution to the fixed point problem

$$
\begin{aligned}
X = \mathcal{H}\Big(&\beta(I-2\mathcal{K})\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)X - \beta^2(\mathcal{K}^2 - \mathcal{K} + 2[\xi\partial_\xi, \mathcal{K}])X \\
&+ \lambda^{-2}\mathcal{F}R(N_{2k-1}(R^{-1}\mathcal{F}^{-1}X) + e_{2k-1})\Big)
\end{aligned}
\tag{6.3}
$$

Remark 6.1. One can also freely add $Ce^{-|\xi_d|^{\frac{1}{2}}\tau}$ to the first component x_d of X. Thus the fixed point argument yields in effect a one parameter family of solutions X depending on the parameter C.

The mapping properties of H are described in the following result, which was proven in [10].

Proposition 6.2. *For any $\alpha \geq 0$ there exists some (large) constant $C = C(\alpha)$ so that the operator $H(\tau,\sigma)$ satisfies the bounds*

$$\|H(\tau,\sigma)\|_{L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho} \lesssim \tau\Big(\frac{\sigma}{\tau}\Big)^C \tag{6.4}$$

$$\Big\|\Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)H(\tau,\sigma)\Big\|_{L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho} \lesssim \Big(\frac{\sigma}{\tau}\Big)^C \tag{6.5}$$

uniformly in $\sigma \geq \tau$.

This leads us to introduce the spaces $L^{\infty,N}L^{2,\alpha}_\rho$ with norm

$$\|f\|_{L^{\infty,N}L^{2,\alpha}_\rho} := \sup_{\tau\geq 1} \tau^N \|f(\tau)\|_{L^{2,\alpha}_\rho}$$

Then the above proposition immediately allows us to draw the following conclusions:

Corollary 6.3. *Given $\alpha \geq 0$, let N be large enough. Then*

$$\|Hb\|_{L^{\infty,N-2}L^{2,\alpha+1/2}_\rho} + \left\|\Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)Hb\right\|_{L^{\infty,N-1}L^{2,\alpha}_\rho} \leq C_0\,\frac{1}{N}\|b\|_{L^{\infty,N}L^{2,\alpha}_\rho}$$

with a constant C_0 that depends on α but does not depend on N.

The small factor N^{-1} is crucial here for our argument to work. For H_0 we have a stronger straightforward counterpart of the above result:

Lemma 6.4. *The operator H_0 satisfies the bounds*

$$\|H_0 b_d\|_{L^{\infty,N}} + \|\partial_\tau H_0 b_d\|_{L^{\infty,N}} \leq C_N \|b_d\|_{L^{\infty,N}}$$

We note that while the constant on the right cannot be small, we no longer lose powers of τ compared to the bounds for H. Hence for fixed N we can choose τ_0 depending on N so that we gain the smallness in the range $\tau > \tau_0$. Combining the last two results we obtain

Proposition 6.5. *Given $\alpha \geq 0$, let N be large enough. Then there exists τ_0 depending on N so that for $\tau > \tau_0$ we have*

$$\|\mathcal{H}b\|_{L^{\infty,N-2}L^{2,\alpha+1/2}_\rho} + \left\|\Big(\partial_\tau - 2\beta(\tau)\xi\partial_\xi\Big)\mathcal{H}b\right\|_{L^{\infty,N-1}L^{2,\alpha}_\rho} \leq C_0\,\frac{1}{N}\|b\|_{L^{\infty,N}L^{2,\alpha}_\rho}$$

with a constant C_0 that depends on α but does not depend on N.

On the other hand, the nonlinear operator N_{2k-1} from (6.1) has the mapping properties stated in Proposition 6.7 below. We first relate the spaces $L^{2,\alpha}_\rho$ to the Sobolev spaces in $\mathbb{R}^3$.

Lemma 6.6. *Let $\alpha \geq 0$. Then*

$$\|x\|_{L^{2,\alpha}_\rho} \asymp \|R^{-1}\mathcal{F}^{-1}x\|_{H^{2\alpha}(\mathbb{R}^3)}$$

Proof. For integer k we have

$$\begin{aligned}\|x\|_{L^{2,k}_\rho} &\asymp \sum_{j=0}^k \|\mathcal{L}^j\mathcal{F}^{-1}x\|_{L^2} \asymp \sum_{j=0}^k \|R^{-1}\mathcal{L}^j\mathcal{F}^{-1}x\|_{L^2(\mathbb{R}^3)}\\ &= \sum_{j=0}^k \|(R^{-1}\mathcal{L}R)^j R^{-1}\mathcal{F}^{-1}x\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

But

$$R^{-1}\mathcal{L}R = -\partial_R^2 - \frac{2}{R}\partial_R - 5W^4(R)$$

where the first two terms can be recognized as the radial part of the three-dimensional Laplacian. Hence we get

$$\|x\|_{L^{2,k}_\rho} \asymp \sum_{j=0}^k \|(-\Delta - 5W^4(R))^j R^{-1}\mathcal{F}^{-1}x\|_{L^2(\mathbb{R}^3)}$$

Since W is bounded together with all its derivatives, the conclusion of the lemma follows for integer α.

For noninteger α we use interpolation. First we consider the map

$$x \mapsto R^{-1}\mathcal{F}^{-1}x$$

and obtain the bound

$$\|R^{-1}\mathcal{F}^{-1}x\|_{H^{2\alpha}(\mathbb{R}^3)} \lesssim \|x\|_{L^{2,\alpha}_\rho}$$

To obtain the reverse bound we use the map

$$u \mapsto \mathcal{F}RS(u)$$

where $S(u)$ stands for the spherical average of a function u in $\mathbb{R}^3$. ☐

Proposition 6.7. *Assume that N is large enough and $\frac{1}{8} \le \alpha < \frac{\nu}{4}$. Then the map*

$$x \to \lambda^{-2}\mathcal{F}R(N_{2k-1}(R^{-1}\mathcal{F}^{-1}x))$$

is locally Lipschitz from $L^{\infty,N-2}L^{2,\alpha+1/2}_\rho$ to $L^{\infty,N}L^{2,\alpha}_\rho$.

Proof. Using the lemma, it remains to prove that the map

$$v \mapsto \lambda^{-2}N_{2k-1}(v)$$

is locally Lipschitz from $L^{\infty,N-2}H^{2\alpha+1}$ to $L^{\infty,N}H^{2\alpha}$. We recall that

$$N_{2k-1}(v) = 5(u_{2k-1}^4 - u_0^4)\,v + 10u_{2k-1}^3\,v^2 + 10u_{2k-1}^2\,v^3 + 5u_{2k-1}\,v^4 + v^5,$$

see the comment following (3.1). The time decay is trivially obtained for all but the first term, for which we need an additional step, where we pull out a factor of $u_{2k-1} - u_0$. Using the regularity of u_{2k-1} given in Theorem 2.1 we obtain

$$u_{2k-1} - u_0 \in \frac{\lambda^{\frac{1}{2}}}{(t\lambda)^2}\,\mathrm{IS}^2(R, \mathcal{Q}_{k-1}), \quad \lambda^{-2}(u_0^4 - u_{2k-1}^4) \in \tau^{-2}S^2(R^{-2}, \mathcal{Q}_{k-1})$$

This indicates that two units of decay in τ are gained. On the level of Sobolev spaces we argue as follows: since we are working with inhomogeneous Sobolev spaces, we can localize the above estimate to unit cubes, as the ℓ^2 summability for $N_{2k-1}(v)$ with respect to unit cubes is inherited from any of the v factors. But in any unit cube Q the coefficients u_{2k-1} have at most $(1-a)^{\frac{\nu+1}{2}-}$ singularities (where $a = \frac{r}{t} \sim \frac{R}{\tau}$) so that we can bound them in Sobolev spaces

$$\|u_{2k-1}\|_{H^{1+2\alpha}(Q)} \lesssim \frac{\lambda^{\frac{1}{2}}}{\mathrm{dist}(Q,0)}, \qquad \|u_{2k-1} - u_0\|_{H^{1+2\alpha}(Q)} \lesssim \frac{\lambda^{\frac{1}{2}}}{\tau^2}\mathrm{dist}(Q,0)$$

where we used that $\alpha < \frac{\nu}{4}$. Then it suffices to establish the quintilinear estimate

$$H^{2\alpha+1} \cdot H^{2\alpha+1} \cdot H^{2\alpha+1} \cdot H^{2\alpha+1} \cdot H^{2\alpha+1} \subset H^{2\alpha}$$

which in three space dimensions holds for $\alpha \ge \frac{1}{8}$ (a standard application of the fractional Leibniz rule and Sobolev imbedding, see [22], page 105). ☐

7. Conclusion

We now prove Theorem 1.1. We first construct a blow-up solution inside the cone as follows. We begin with the approximate solution u_{2k-1} and the error e_{2k-1} given by Theorem 2.1 inside the cone. We extend them outside the cone to functions having the same size and regularity, supported in $r < 2t$. Then the relation

$$e_{2k-1} = (-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u_{2k-1} + u_{2k-1}^5$$

is valid only inside the cone.

Using Propositions 6.5, 6.7 we iteratively find a solution

$$X \in L^{\infty,N-2} L_\rho^{2,\alpha+1/2}$$

for the equation (6.3) for $t \le t_0$ sufficiently small and

$$\frac{1}{8} \le \alpha < \frac{\nu}{4}$$

Then we set

$$v = R^{-1}\mathcal{F}^{-1}X \in L^{\infty,N-2}H^{2\alpha+1}$$

and

$$u = u_{2k-1} + v$$

Given the derivation of (6.3), the function u solves

$$(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u + u^5 = (-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u_{2k-1} + u_{2k-1}^5 - e_{2k-1}$$

which implies that the function u solves the nonlinear wave equation

$$(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)u + u^5 = 0$$

inside the cone.

The second part of the argument is to extend the above solution u to the exterior of the cone $K = \{0 < t < t_0,\ 0 \le |x| \le t\}$ so that the blow-up occurs only at the tip of the cone. For this we first observe that the above function u is close to u_0 inside the cone and close to 0 outside, namely

$$\lim_{t\to 0} \int_{K_t} |\nabla(u(t) - u_0(t))|^2 + |u(t) - u_0(t)|^6\, dx = 0$$

and

$$\lim_{t\to 0} \int_{K_t^c} |\nabla u(t)|^2 + |u(t)|^6\, dx = 0$$

Hence given $\delta > 0$ we can choose t_0 so that the two quantities above are less than δ^6.

We let w be the solution to the equation

$$(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)w + w^5 = 0$$

with initial data

$$w(t_0) = u(t_0), \qquad w_t(t_0) = u_t(t_0)$$

Due to the finite speed of propagation we conclude that $w = u$ inside the cone. To conclude the proof of the theorem we will show that w cannot blow up outside the cone before time 0. For this it suffices to prove that the energy outside the cone stays small,

$$\int_{K_t^c} |\nabla w(t)|^2 + |w(t)|^6 \, dx \lesssim \delta \tag{7.1}$$

see [17], [18].

This is proved using energy conservation. The energy $\mathcal{E}(w(t))$ is conserved in time. At time t_0 we have

$$\mathcal{E}(w(t_0)) = \mathcal{E}(u(t_0)) = \mathcal{E}(W) + O(\delta)$$

hence at time t we must have a similar relation. But the energy inside the cone is already close to this, so we obtain

$$\left| \int_{K_t^c} \frac{1}{2}(u_t^2 + |\nabla u|)^2 - \frac{|u|^6}{6} \, dx \right| \lesssim \delta$$

On the other hand, we have the Sobolev inequality

$$\int_{K_t^c} |u|^6 dx \lesssim \left(\int_{K_t^c} |\nabla u|^2 \, dx \right)^3$$

with a universal constant (independent of t). Combining the two inequalities above we see that for each t there are two possibilities. Either we have

$$\int_{K_t^c} \frac{1}{2}(u_t^2 + |\nabla u|)^2 \, dx \lesssim \delta$$

or

$$\int_{K_t^c} \frac{1}{2}(u_t^2 + |\nabla u|)^2 \, dx \gtrsim 1$$

The first alternative holds at $t = t_0$. Then a continuity argument shows that it must hold at all t, since the above integral is a continuous function of t for as long as it stays small.

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